COMPUTING α-INVARIANTS OF SINGULAR DEL PEZZO SURFACES

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Abstract. We prove new local inequality for divisors on surfaces and utilize it to compute α-invariants of singular del Pezzo surfaces, which implies that del Pezzo surfaces of degree one whose singular points are of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_5$ or $\mathbb{A}_6$ are Kähler-Einstein.

We assume that all varieties are projective, normal, and defined over $\mathbb{C}$.

1. Introduction

Let X be a Fano variety with at most quotient singularities (a Fano orbifold).

Theorem 1.1 ([37])**.** If $\dim(X)=2$ and X is smooth, then

$$\text{the surface } X \text{ is Kähler–Einstein} \iff \text{the group } \mathrm{Aut}\big(X\big) \text{ is reductive.}$$

An important role in the proof of Theorem 1.1 is played by several holomorphic invariants, which are now known as α-invariants. Let us describe their algebraic counterparts.

Let D be an effective $\mathbb{Q}$-divisor on the variety X. Then the number

$$\mathrm{c}\big(X,D\big)=\sup\Big\{\epsilon\in\mathbb{Q}\ \Big|\text{the log pair }\big(X,\epsilon D\big)\text{ is log canonical}\Big\}\in\mathbb{Q}\cup\big\{+\infty\big\}.$$

is called the log canonical threshold of the divisor D (see [21, Definition 8.1]). Put

$$\mathrm{lct}_n\big(X\big)=\inf\left\{\mathrm{c}\left(X,\frac{1}{n}B\right)\ \middle|\ B\text{ is a divisor in }\big|-nK_X\big|\right\}$$

for every $n\in\mathbb{N}$. For small n, the number $\mathrm{lct}_n(X)$ is usually not very hard to compute.

Example 1.2 ([28])**.** If X is a smooth surface in $\mathbb{P}^3$ of degree 3, then

$$\mathrm{lct}_1\big(X\big)=\begin{cases}2/3 \text{ if } X \text{ has an Eckardt point,}\\ 3/4 \text{ if } X \text{ has no Eckardt points.}\end{cases}$$

The number $\mathrm{lct}_n(X)$ is denoted by $\alpha_n(X)$ in [38].

Remark 1.3. It follows from [27, Lemma 4.8] that the set

$$\left\{\mathrm{c}\left(X,\frac{1}{n}B\right)\ \middle|\ B\text{ is a divisor in }\big|-nK_X\big|\right\}$$

is finite (cf. [23]). Thus, there exists a divisor $B\in|-nK_X|$ such that $\mathrm{lct}_n(X)=\mathrm{c}(X,B/n)\in\mathbb{Q}$.

If the variety X is smooth, then it is proved by Demailly (see [6, Theorem A.3]) that

$$\inf\Big\{\mathrm{lct}_n\big(X\big)\ \Big|\ n\in\mathbb{N}\Big\}=\alpha\big(X\big),$$

where $\alpha(X)$ is the α-invariant introduced by Tian in [36]. Put $\mathrm{lct}(X)=\inf\{\mathrm{lct}_n(X)\mid n\in\mathbb{N}\}$.

Conjecture 1.4 ([38, Question 1])**.** There is an $n\in\mathbb{N}$ such that $\mathrm{lct}(X)=\mathrm{lct}_n(X)$.

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The proof of Theorem 1.1 uses (at least implicitly) the following result.

Theorem 1.5 ([36], [10])**.** The Fano orbifold X is Kähler–Einstein if

$$\mathrm{lct}\big(X\big) > \frac{\dim(X)}{\dim(X)+1}.$$

Note that there are many well-known obstructions to the existence of Kähler–Einstein metrics on smooth Fano manifolds and Fano orbifolds (see [25], [14], [15], [34]).

Example 1.6. If $X \cong \mathbb{P}(1,2,3)$, then X is not Kähler–Einstein (see [15], [34]).

Let us describe one more α-invariant that took its origin in [37].

Let $\mathcal{M}$ be a linear system on the variety X. Then the number

$$\mathrm{c}\big(X,\mathcal{M}\big) = \sup\Big\{\epsilon \in \mathbb{Q}\ \Big|\ \text{the log pair } \big(X,\epsilon\mathcal{M}\big) \text{ is log canonical}\Big\} \in \mathbb{Q}\cup\big\{+\infty\big\}.$$

is called the log canonical threshold of the linear system $\mathcal{M}$ (cf. [21, Theorem 4.8]). Put

$$\mathrm{lct}_{n,2}\big(X\big) = \inf\left\{\mathrm{c}\left(X,\frac{1}{n}\mathcal{B}\right)\ \middle|\ \mathcal{B} \text{ is a pencil in } \big|-nK_X\big|\right\}$$

for every $n\in\mathbb{N}$. The number $\mathrm{lct}_{n,2}(X)$ is denoted by $\alpha_{n,2}(X)$ in [8] and [41]. Note that

$$\mathrm{lct}\big(X\big) = \inf\Big\{\mathrm{lct}_{n,2}\big(X\big)\ \Big|\ n\in\mathbb{N}\Big\}, \tag{1.7}$$

and it follows from [21, Theorem 4.8] that $\mathrm{lct}_n(X) \leqslant \mathrm{lct}_{n,2}(X)$ for every $n\in\mathbb{N}$.

Remark 1.8. It follows from [27, Lemma 4.8] and [21, Theorem 4.8] that the set

$$\left\{\mathrm{c}\left(X,\frac{1}{n}\mathcal{B}\right)\ \middle|\ \mathcal{B} \text{ is a pencil in } \big|-nK_X\big|\right\}$$

is finite. Thus, there is a pencil $\mathcal{B}$ in $|-nK_X|$ such that the equality $\mathrm{lct}_{n,2}(X)=\mathrm{c}(X,\mathcal{B}/n)$. Then

$$\mathrm{lct}_{n,2}\big(X\big) > \mathrm{lct}\big(X\big)$$

if there exists at most finitely many effective $\mathbb{Q}$-divisors $D_1, D_2,\ldots,D_r$ on the variety X such that

$$\mathrm{c}\big(X,D_1\big)=\mathrm{c}\big(X,D_2\big)=\cdots=\mathrm{c}\big(X,D_r\big)=\mathrm{lct}\big(X\big)$$

and $D_1\sim_{\mathbb{Q}} D_2\sim_{\mathbb{Q}}\ldots\sim_{\mathbb{Q}} D_r\sim_{\mathbb{Q}} -K_X$.

The importance of the number $\mathrm{lct}_{n,2}(X)$ is due to the following conjecture.

Conjecture 1.9 (cf. [8, Theorem 2], [41, Theorem 1])**.** Suppose that

$$\mathrm{lct}_{n,2}\big(X\big) > \frac{\dim(X)}{\dim(X)+1}.$$

for every $n\in\mathbb{N}$. Then X is Kähler–Einstein.

Note that Conjecture 1.9 is not much stronger than Theorem 1.5 by (1.7).

Example 1.10. Suppose that X is a smooth hypersurface in $\mathbb{P}^m$ of degree $m\geqslant 3$. Then

$$\mathrm{lct}_n\big(X\big) \geqslant 1-\frac{1}{m} = \frac{\dim(X)}{\dim(X)+1}$$

for every $n\in\mathbb{N}$ by [2]. The equality $\mathrm{lct}_n(X)=1-1/m$ holds $\iff$ the hypersurface X contains a cone of dimension $m-2$ (see [2, Theorem 1.3], [2, Theorem 4.1], [13, Theorem 0.2]). Then

$$\mathrm{lct}_{n,2}\big(X\big) > \frac{\dim(X)}{\dim(X)+1}$$

by Remark 1.8, [2, Remark 1.6], [2, Theorem 4.1], [2, Theorem 5.2] and [13, Theorem 0.2], because X contains at most finitely many cones by [9, Theorem 4.2]. If X is general, then

$$1 = \mathrm{lct}_1\big(X\big) \geqslant \mathrm{lct}\big(X\big) \geqslant \begin{cases} 3/4 \text{ if } m = 3, \\ 7/9 \text{ if } m = 4, \\ 5/6 \text{ if } m = 5, \\ 1 \text{ if } m \geqslant 5, \end{cases}$$

by [33], [3], [5]. Thus, if X is general, then it is Kähler–Eisntein by Theorem 1.5.

The assertion of Conjecture 1.9 follows from [8, Theorem 2] and [41, Theorem 1] under an additional assumption that the Kähler-Ricci flow on X is tamed (see [8] and [41]).

Theorem 1.11 ([8], [41])**.** If $\dim(X) = 2$, then the Kähler-Ricci flow on X is tamed.

Corollary 1.12. Suppose that $\dim(X) = 2$ and

$$\mathrm{lct}_{n,2}\big(X\big) > \frac{2}{3}$$

for every $n \in \mathbb{N}$. Then X is Kähler–Einstein.

Two-dimensional Fano orbifolds are called del Pezzo surfaces.

Remark 1.13. Del Pezzo surfaces with quotient singularities are not classified (cf. [20]). But

- del Pezzo surfaces with canonical singularities are classified (see [18]),
- del Pezzo surfaces with 2-Gorenstein quotient singularities are classified (see [1]),
- del Pezzo surfaces of Picard rank 1 with T-singularities are classified (see [17]).

Del Pezzo surfaces with canonical singularities form a very natural class of del Pezzo surfaces.

Problem 1.14. Describe all Kähler–Einstein del Pezzo surface with canonical singularities.

Recall that if X is a del Pezzo surface with canonical singularities, then

- either the inequality $K_X^2 \geqslant 5$ holds,
- or one of the following possible cases occurs:
  - the equality $K_X^2 = 1$ holds and X is a sextic surface in $\mathbb{P}(1,1,2,3)$,
  - the equality $K_X^2 = 2$ holds and X is a quartic surface in $\mathbb{P}(1,1,1,2)$,
  - the equality $K_X^2 = 3$ holds and X is a cubic surface in $\mathbb{P}^3$,
  - the equality $K_X^2 = 4$ holds and X is a complete intersection in $\mathbb{P}^4$ of two quadrics.

Let us consider few examples to illustrate the expected answer to Problem 1.14.

Example 1.15. Suppose that X is a sextic surface in $\mathbb{P}(1,1,2,3)$ such that its singular locus consists of singular points of type $\mathbb{A}_1$ or $\mathbb{A}_2$. Arguing as in the proof of [3, Lemma 4.1], we see that

$$\mathrm{lct}_{n,2}\big(X\big) > \frac{2}{3}$$

for every $n \in \mathbb{N}$. Thus, the surface X is Kähler–Einstein by Corollary 1.12.

Example 1.16. Suppose that X is a quartic surface in $\mathbb{P}(1,1,1,2)$ such that its singular locus consists of singular points of type $\mathbb{A}_1$ or $\mathbb{A}_2$. Then X is Kähler–Einstein by [16, Theorem 2].

Example 1.17. Suppose that X is a cubic surface in $\mathbb{P}^3$ that is not a cone. Then

- if X is smooth, then X is Kähler–Einstein by Theorem 1.1,
- if $\mathrm{Sing}(X)$ consists of one point of type $\mathbb{A}_1$, then it follows from [35, Theorem 5.1] that

  $$\mathrm{lct}_{n,2}\big(X\big) > \frac{2}{3} = \mathrm{lct}_1\big(X\big) = \mathrm{lct}\big(X\big)$$

  for every $n \in \mathbb{N}$, which implies that X is Kähler–Einstein by Corollary 1.12,

- if the cubic surface X has a singular point that is not a singular point of type $\mathbb{A}_1$ or $\mathbb{A}_2$, then the surface X is not Kähler–Einstein by [11, Proposition 4.2].

Example 1.18. Suppose that X is a complete intersection in $\mathbb{P}^4$ of two quadrics. Then

- if X is smooth, then X is Kähler–Einstein by Theorem 1.1,
- if X is Kähler–Einstein, then X has at most singular points of type $\mathbb{A}_1$ (see [19]),
- it follows from [24] or [16, Theorem 44] that X is Kähler–Einstein if it is given by
$$\sum_{i=0}^{4} x_i^2 = \sum_{i=0}^{4} \lambda_i x_i^2 = 0 \subseteq \mathbb{P}^4 \cong \mathrm{Proj}\Big(\mathbb{C}[x_0, \ldots, x_4]\Big),$$
and X has at most singular points of type $\mathbb{A}_1$, where $(\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4) \in \mathbb{P}^4$.

Keeping in mind Examples 1.15, 1.16, 1.17 and 1.18, [4, Example 1.12] and [26, Table 1], it is very natural to expect that the following answer to Problem 1.14 is true (cf. Example 1.6).

Conjecture 1.19. If the orbifold X is a del Pezzo surface with at most canonical singularities, then the surface X is Kähler–Enstein $\iff$ it satisfies one of the following conditions:

- $K_X^2 = 1$ and $\mathrm{Sing}(X)$ consists of points of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_5$, $\mathbb{A}_6$, $\mathbb{A}_7$ or $\mathbb{D}_4$,
- $K_X^2 = 2$ and $\mathrm{Sing}(X)$ consists of points of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$,
- $K_X^2 = 3$ and $\mathrm{Sing}(X)$ consists of points of type $\mathbb{A}_1$ or $\mathbb{A}_2$,
- $K_X^2 = 4$ and $\mathrm{Sing}(X)$ consists of points of type $\mathbb{A}_1$,
- the surface X is smooth and $6 \geqslant K_X^2 \geqslant 5$,
- either $X \cong \mathbb{P}^2$ or $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.

In this paper, we prove the following result.

Theorem 1.20. Suppose that X is a sextic surface in $\mathbb{P}(1,1,2,3)$. Then
$$\mathrm{lct}_{n,2}\big(X\big) > \frac{2}{3}$$
for every $n \in \mathbb{N}$ if $\mathrm{Sing}(X)$ consists of points of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_5$ or $\mathbb{A}_6$.

Corollary 1.21. Suppose that X is a sextic surface in $\mathbb{P}(1,1,2,3)$ such that its singular locus consists of singular points of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_5$ or $\mathbb{A}_6$. Then X is Kähler–Enstein.

It should be pointed out that Corollary 1.21 and Examples 1.15, 1.16, 1.17, 1.18 illustrate a general philosophy that the existence of Kähler–Enstein metrics on Fano orbifolds is related to an algebro-geometric notion of stability (see [11, Theorem 4.1], [39], [12]).

Remark 1.22. If X is a sextic surface in $\mathbb{P}(1,1,2,3)$ with canonical singularities, then either
$$\mathrm{Sing}\big(X\big) \in \left\{\begin{array}{l} \mathbb{E}_8, \mathbb{E}_7, \mathbb{E}_7+\mathbb{A}_1, \mathbb{E}_6, \mathbb{E}_6+\mathbb{A}_2, \mathbb{E}_6+\mathbb{A}_1, \mathbb{D}_8, \mathbb{D}_7, \mathbb{D}_6, \mathbb{D}_6+\mathbb{A}_1+\mathbb{A}_1, \mathbb{D}_6+\mathbb{A}_1,\\ \mathbb{D}_5, \mathbb{D}_5+\mathbb{A}_3, \mathbb{D}_5+\mathbb{A}_2, \mathbb{D}_5+\mathbb{A}_1+\mathbb{A}_1, \mathbb{D}_5+\mathbb{A}_1, \mathbb{D}_4, \mathbb{D}_4+\mathbb{D}_4, \mathbb{D}_4+\mathbb{A}_3, \mathbb{D}_4+\mathbb{A}_2,\\ \mathbb{D}_4+\mathbb{A}_1+\mathbb{A}_1+\mathbb{A}_1+\mathbb{A}_1, \mathbb{D}_4+\mathbb{A}_1+\mathbb{A}_1+\mathbb{A}_1, \mathbb{D}_4+\mathbb{A}_1+\mathbb{A}_1, \mathbb{D}_4+\mathbb{A}_1, \mathbb{A}_8,\\ \mathbb{A}_7, \mathbb{A}_7+\mathbb{A}_1, \mathbb{A}_6, \mathbb{A}_6+\mathbb{A}_1, \mathbb{A}_5, \mathbb{A}_5+\mathbb{A}_1, \mathbb{A}_5+\mathbb{A}_1+\mathbb{A}_1, \mathbb{A}_5+\mathbb{A}_2, \mathbb{A}_5+\mathbb{A}_2+\mathbb{A}_1,\\ \mathbb{A}_4, \mathbb{A}_4+\mathbb{A}_4, \mathbb{A}_4+\mathbb{A}_3, \mathbb{A}_4+\mathbb{A}_2+\mathbb{A}_1, \mathbb{A}_4+\mathbb{A}_2, \mathbb{A}_4+\mathbb{A}_1+\mathbb{A}_1, \mathbb{A}_4+\mathbb{A}_1,\\ \mathbb{A}_3, \mathbb{A}_3+\mathbb{A}_3, \mathbb{A}_3+\mathbb{A}_3+\mathbb{A}_1+\mathbb{A}_1, \mathbb{A}_3+\mathbb{A}_2, \mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1, \mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1+\mathbb{A}_1,\\ \mathbb{A}_3+\mathbb{A}_1+\mathbb{A}_1+\mathbb{A}_1+\mathbb{A}_1, \mathbb{A}_3+\mathbb{A}_1+\mathbb{A}_1+\mathbb{A}_1, \mathbb{A}_3+\mathbb{A}_1+\mathbb{A}_1, \mathbb{A}_3+\mathbb{A}_1 \end{array}\right\}$$
or $\mathrm{Sing}(X)$ consists only of points of type $\mathbb{A}_1$ and $\mathbb{A}_2$ (see [40]).

What is known about α-invariants of del Pezzo surfaces with canonical singularities?

Theorem 1.23 ([3])**.** If X is a smooth del Pezzo surface, then $\mathrm{lct}(X) = \mathrm{lct}_1(X)$.

Theorem 1.24 ([3], [31])**.** If X is a del Pezzo surface with canonical singularities, then

$$\mathrm{lct}\big(X\big)=\mathrm{lct}_1\big(X\big)$$

in the case when $K_X^2\geqslant 3$.

Theorem 1.25 ([31])**.** If X is a quartic surface in $\mathbb{P}(1,1,1,2)$ with canonical singularities, then

$$\mathrm{lct}\big(X\big)=\begin{cases}\mathrm{lct}_2\big(X\big)=1/3 \text{ if } X \text{ has a singular point of type } \mathbb{A}_7,\\ \mathrm{lct}_2\big(X\big)=2/5 \text{ if } X \text{ has a singular point of type } \mathbb{A}_6,\\ \mathrm{lct}_1\big(X\big) \text{ in the remaining cases.}\end{cases}$$

In this paper, we prove the following result (cf. Example 1.15).

Theorem 1.26. Suppose that X is a sextic surface in $\mathbb{P}(1,1,2,3)$ with canonical singularities, let $\omega\colon X\to\mathbb{P}(1,1,2)$ be a natural double cover, and let R be its branch curve in $\mathbb{P}(1,1,2)$. Then

$$\mathrm{lct}\big(X\big)=\begin{cases}\mathrm{lct}_2\big(X\big)=1/3 \text{ if } \mathrm{Sing}(X) \text{ consists of a point of type } \mathbb{D}_8,\\ \mathrm{lct}_2\big(X\big)=2/5 \text{ if } \mathrm{Sing}(X) \text{ consists of a point of type } \mathbb{D}_7,\\ \mathrm{lct}_3\big(X\big)=1/2 \text{ if } \mathrm{Sing}(X) \text{ consists of a point of type } \mathbb{A}_8,\\ \mathrm{lct}_2\big(X\big)=1/2 \text{ if } \mathrm{Sing}(X) \text{ consists of a point of type } \mathbb{A}_7 \text{ and a point of type } \mathbb{A}_1,\\ \mathrm{lct}_2\big(X\big)=1/2 \text{ if } \mathrm{Sing}(X) \text{ consists of a point of type } \mathbb{A}_7 \text{ and } R \text{ is reducible},\\ \mathrm{lct}_3\big(X\big)=3/5 \text{ if } X \text{ has a singular point of type } \mathbb{A}_7 \text{ and } R \text{ is irreducible},\\ \mathrm{lct}_2\big(X\big)=2/3 \text{ if } X \text{ has a singular point of type } \mathbb{A}_6,\\ \mathrm{lct}_2\big(X\big)=2/3 \text{ if } X \text{ has a singular point of type } \mathbb{A}_5,\\ \mathrm{lct}_2\big(X\big)=\min\big(\mathrm{lct}_1\big(X\big),4/5\big) \text{ if } X \text{ has a singular point of type } \mathbb{A}_4,\\ \mathrm{lct}_1\big(X\big) \text{ in the remaining cases.}\end{cases}$$

It should be pointed out that if X is a del Pezzo surface with at most canonical singularities, then all possible values of the number $\mathrm{lct}_1(X)$ are computed in [28], [29], [30].

Example 1.27. If X is a sextic surface in $\mathbb{P}(1,1,2,3)$ with canonical singularities, then

- $\mathrm{lct}_1(X)=1/6 \iff$ the surface X has a singular point of type $\mathbb{E}_8$,
- $\mathrm{lct}_1(X)=1/4 \iff$ the surface X has a singular point of type $\mathbb{E}_7$,
- $\mathrm{lct}_1(X)=1/3 \iff$ the surface X has a singular point of type $\mathbb{E}_6$,
- $\mathrm{lct}_1(X)=1/2 \iff$ the surface X has a singular point of type $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{D}_6$, $\mathbb{D}_7$ or $\mathbb{D}_8$,
- $\mathrm{lct}_1(X)=2/3 \iff$ the following two conditions are satisfied:
  - the surface X has no singular points of type $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{D}_6$, $\mathbb{D}_7$, $\mathbb{D}_8$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$,
  - there is a curve in $|-K_X|$ that has a cusp at a point in $\mathrm{Sing}(X)$ of type $\mathbb{A}_2$,
- $\mathrm{lct}_1(X)=3/4 \iff$ the following three conditions are satisfied:
  - the surface X has no singular points of type $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{D}_6$, $\mathbb{D}_7$, $\mathbb{D}_8$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$,
  - there is no curve in $|-K_X|$ that has a cusp at a point in $\mathrm{Sing}(X)$ of type $\mathbb{A}_2$,
  - there is a curve in $|-K_X|$ that has a cusp at a point in $\mathrm{Sing}(X)$ of type $\mathbb{A}_1$,
- $\mathrm{lct}_1(X)=5/6 \iff$ the following three conditions are satisfied:
  - the surface X has no singular points of type $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{D}_6$, $\mathbb{D}_7$, $\mathbb{D}_8$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$,
  - there is no curve in $|-K_X|$ that have a cusp at a point in $\mathrm{Sing}(X)$,
  - there is a curve in $|-K_X|$ that has a cusp,
- $\mathrm{lct}_1(X)=1 \iff$ there are no cuspidal curves in $|-K_X|$.

A crucial role in the proofs of both Theorems 1.26 and 1.20 is played by a new local inequality that we discovered. This inequality is a technical tool, but let us describe it now.

Let S be a surface, let D be an arbitrary effective $\mathbb{Q}$-divisor on the surface S, let O be a smooth point of the surface S, let Δ_1 and Δ_2 be reduced irreducible curves on S such that

$$\Delta_1 \not\subseteq \mathrm{Supp}\big(D\big) \not\supseteq \Delta_2,$$

and the divisor $\Delta_1+\Delta_2$ has a simple normal crossing singularity at the smooth point $O \in \Delta_1\cap\Delta_2$, let a_1 and a_2 be some non-negative rational numbers. Suppose that the log pair

$$\Big(S,\ D+a_1\Delta_1+a_2\Delta_2\Big)$$

is not Kawamata log terminal at O, but $(S, D+a_1\Delta_1+a_2\Delta_2)$ is Kawamata log terminal in a punctured neighborhood of the point O.

Theorem 1.28. Let $A, B, M, N, \alpha, \beta$ be non-negative rational numbers. Then

$$\mathrm{mult}_O\Big(D\cdot\Delta_1\Big)\geqslant M+Aa_1-a_2 \text{ or } \mathrm{mult}_O\Big(D\cdot\Delta_2\Big)\geqslant N+Ba_2-a_1$$

in the case when the following conditions are satisfied:

- the inequality $\alpha a_1+\beta a_2\leqslant 1$ holds,
- the inequalities $A(B-1)\geqslant 1\geqslant \max(M,N)$ hold,
- the inequalities $\alpha(A+M-1)\geqslant A^2(B+N-1)\beta$ and $\alpha(1-M)+A\beta\geqslant A$ hold,
- either the inequality $2M+AN\leqslant 2$ holds or

$$\alpha\big(B+1-MB-N\big)+\beta\big(A+1-AN-M\big)\geqslant AB-1.$$

Corollary 1.29. Suppose that

$$\frac{2m-2}{m+1}a_1+\frac{2}{m+1}a_2\leqslant 1$$

for some integer m such that $m\geqslant 3$. Then

$$\mathrm{mult}_O\Big(D\cdot\Delta_1\Big)\geqslant 2a_1-a_2 \text{ or } \mathrm{mult}_O\Big(D\cdot\Delta_2\Big)\geqslant \frac{m}{m-1}a_2-a_1.$$

Proof. To prove the required assertion, let us put

$$A=2,\ B=\frac{m}{m-1},\ M=0, N=0,\ \alpha=\frac{2m-2}{m+1},\ \beta=\frac{2}{m+1}a_2,$$

and let us check that all hypotheses of Theorem 1.28 are satisfied.

We have $\alpha a_1+\beta a_2\leqslant 1$ by assumption. We have

$$A(B-1)=\frac{2}{m-1}\geqslant 1\geqslant 0=\max(M,N),$$

since $m\geqslant 3$. We have

$$\alpha(A+M-1)=\frac{2m-2}{m+1}\geqslant\frac{8}{m^2-1}=A^2(B+N-1)\beta,$$

since $m\geqslant 3$. We have $\alpha(1-M)+A\beta=2\geqslant 2=A$ and $2M+AN=0\leqslant 2$.

Thus, we see that all hypotheses of Theorem 1.28 are satisfied. Then

$$\mathrm{mult}_O\Big(D\cdot\Delta_1\Big)\geqslant M+Aa_1-a_2=2a_1-a_2 \text{ or } \mathrm{mult}_O\Big(D\cdot\Delta_2\Big)\geqslant N+Ba_2-a_1=\frac{m}{m-1}a_2-a_1$$

by Theorem 1.28. □

For the convenience of a reader, we organize the paper in the following way:

- in Section 2, we collect auxiliary results,
- in Section 3, we prove Theorem 1.28,
- in Sections 4, we prove Theorem 4.1,
- in Sections 5, we prove Theorems 5.1,
- in Sections 6, we prove Theorems 6.1.

By Remark 1.22, both Theorems 1.20 and 1.26 follow from Theorems 4.1, 5.1 and 6.1.

2. Preliminaries

Let S be a surface with canonical singularities, and let D be an effective $\mathbb{Q}$-divisor on S. Put

$$D=\sum_{i=1}^{r}a_iD_i,$$

where D_i is an irreducible curve, and $a_i\in\mathbb{Q}_{>0}$. We assume that $D_i\neq D_j \iff i\neq j$.

Suppose that (S,D) is log canonical, but (S,D) is not Kawamata log terminal.

Remark 2.1. Let $\bar{D}$ be an effective $\mathbb{Q}$-divisor on the surface S such that

$$\bar{D}=\sum_{i=1}^{r}\bar{a}_iD_i\sim_{\mathbb{Q}} D,$$

and the log pair $(S,\bar{D})$ is log canonical, where $\bar{a}_i$ is a non-negative rational number. Put

$$\alpha=\min\left\{\frac{a_i}{\bar{a}_i}\ \Big|\ \bar{a}_i\neq 0\right\},$$

where α is well defined and $\alpha\leqslant 1$. Then $\alpha=1\iff D=\bar{D}$. Suppose that $D\neq\bar{D}$. Put

$$D'=\sum_{i=1}^{r}\frac{a_i-\alpha\bar{a}_i}{1-\alpha}D_i,$$

and choose $k\in\{1,\ldots,r\}$ such that $\alpha=a_k/\bar{a}_k$. Then $D_k\not\subset\mathrm{Supp}(D')$ and $D'\sim_{\mathbb{Q}}\bar{D}\sim_{\mathbb{Q}}D$, but the log pair (S,D') is not Kawamata log terminal.

Let $\mathrm{LCS}(S,D)$ be the locus of log canonical singularities of the log pair (S,D) (see [6]).

Theorem 2.2 ([22, Theorem 17.4])**.** If $-(K_S+D)$ is nef and big, then $\mathrm{LCS}(S,D)$ is connected.

Take a point $P\in\mathrm{LCS}(S,D)$. Suppose that $\mathrm{LCS}(S,D)$ contains no curves that pass through P.

Lemma 2.3. Suppose that $P\notin\mathrm{Sing}(S)$ and $P\notin\mathrm{Sing}(D_1)$. Then

$$D_1\cdot\left(\sum_{i=2}^{r}a_iD_i\right)\geqslant\sum_{i=2}^{r}a_i\mathrm{mult}_P\Big(D_1\cdot D_i\Big)>1.$$

Proof. The log pair $(S,D_1+\sum_{i=2}^{r}a_iD_i)$ is not log canonical at P, since $a_1<1$. Then

$$D_1\cdot\sum_{i=2}^{r}a_iD_i\geqslant\sum_{i=2}^{r}a_i\mathrm{mult}_P\Big(D_1\cdot D_i\Big)\geqslant\mathrm{mult}_P\left(\sum_{i=2}^{r}a_iD_i\Big|_{D_1}\right)>1$$

by [22, Theorem 17.6]. □

Let $\pi\colon\bar{S}\to S$ be a birational morphism, and $\bar{D}$ is a proper transform of D via π. Then

$$K_{\bar{S}}+\bar{D}+\sum_{i=1}^{s}e_iE_i\sim_{\mathbb{Q}}\pi^*\big(K_S+D\big),$$

where E_i is an irreducible π-exceptional curve, and $e_i\in\mathbb{Q}$. We assume that $E_i=E_j\iff i=j$.

Suppose, in addition, that the birational morphism π induces an isomorphism

$$\bar{S}\setminus\left(\bigcup_{i=1}^{s}E_i\right)\cong S\setminus P.$$

Remark 2.4. The log pair $(\bar{S},\bar{D}+\sum_{i=1}^{s}e_iE_i)$ is not Kawamata log terminal at a point in $\cup_{i=1}^{s}E_i$.

Suppose that S is singular at P, and either P is a singular point of type $\mathbb{D}_n$ for some $n\in\mathbb{N}_{\geqslant 4}$, or the point P is a singular point of type $\mathbb{E}_m$ for some $m\in\{6,7,8\}$.

Lemma 2.5. Suppose that $E_1^2=E_2^2=\cdots=E_s^2=-2$. Then $e_1=1$ if

$$E_1\cdot\Bigg(\sum_{i=2}^{s}E_i\Bigg)=3.$$

Proof. This follows from [32, Proposition 2.9], because $(S\ni P)$ is a weakly-exceptional singularity (see [32, Example 4.7], [7, Example 3.4], [7, Theorem 3.15]). ☐

Lemma 2.6. Suppose that S is a sextic surface in $\mathbb{P}(1,1,2,3)$ that has canonical singularities, and suppose that $D\sim_{\mathbb{Q}}-K_X$. Let μ be a positive rational number such that either

$$\mu<\mathrm{lct}_1\big(S\big),$$

or $\mu=2/3$ and D is not a curve in $|-K_X|$ with a cusp at a point in $\mathrm{Sing}(S)$ of type $\mathbb{A}_2$. Then

$$\mathrm{LCS}\big(S,\mu D\big)\subseteq\mathrm{Sing}\big(S\big),$$

the locus $\mathrm{LCS}(S,\mu D)$ contains no points of type $\mathbb{A}_1$ or $\mathbb{A}_2$, and $|\mathrm{LCS}(S,\mu D)|\leqslant 1$.

Proof. This follows from Theorem 2.2 and the proof of [3, Lemma 4.1]. ☐

Most of the described results are valid in much more general settings (cf. [22] and [21]).

3. Local inequality

The purpose of this section is to prove Theorem 1.28.

Let S be a surface, let D be an arbitrary effective $\mathbb{Q}$-divisor on the surface S, let O be a smooth point of the surface S, let Δ_1 and Δ_2 be reduced irreducible curves on S such that

$$\Delta_1\not\subseteq\mathrm{Supp}\big(D\big)\not\supseteq\Delta_2,$$

and the divisor $\Delta_1+\Delta_2$ has a simple normal crossing singularity at the smooth point $O\in\Delta_1\cap\Delta_2$, let a_1 and a_2 be some non-negative rational numbers. Suppose that the log pair

$$\Big(S,\ D+a_1\Delta_1+a_2\Delta_2\Big)$$

is not Kawamata log terminal at O, but $(S,D+a_1\Delta_1+a_2\Delta_2)$ is Kawamata log terminal in a punctured neighborhood of the point O. In particular, we must have $a_1<1$ and $a_2<1$.

Let A,B,M,N,α,β be non-negative rational numbers such that

- the inequality $\alpha a_1+\beta a_2\leqslant 1$ holds,
- the inequalities $A(B-1)\geqslant 1\geqslant\max(M,N)$ hold,
- the inequalities $\alpha(A+M-1)\geqslant A^2(B+N-1)\beta$ and $\alpha(1-M)+A\beta\geqslant A$ holds,
- either the inequality $2M+AN\leqslant 2$ holds or

$$\alpha\big(B+1-MB-N\big)+\beta\big(A+1-AN-M\big)\geqslant AB-1.$$

Lemma 3.1. The inequalities $A+M\geqslant 1$ and $B>1$ holds. The inequality

$$\alpha\big(B+1-MB-N\big)+\beta\big(A+1-AN-M\big)\geqslant AB-1$$

holds. The inequality $\beta(1-N)+B\alpha\geqslant B$ holds. The inequalities

$$\frac{\alpha(2-M)}{A+1}+\frac{\beta(2-N)}{B+1}\geqslant 1$$

and $\alpha(2-M)B+\beta(1-N)(A+1)\geqslant B(A+1)$ hold.

Proof. The inequality $B > 1$ follows from the inequality $A(B-1) \geqslant 1$. Then

$$\frac{\alpha}{A+1} + \frac{\beta}{B+1} \geqslant \frac{\alpha}{A+1} + \frac{\beta}{2B} \geqslant \frac{1}{2}$$

because $2B \geqslant B+1$. Similarly, we see that $A+M \geqslant 1$, because

$$\frac{\alpha(A+M-1)}{A^2(B+N-1)} \geqslant \beta \geqslant 0$$

and $B+N-1 \geqslant 0$. The inequality $\beta(1-N) + B\alpha \geqslant B$ follows from the inequalities

$$\alpha + \frac{\beta(1-N)}{B} \geqslant \frac{2-M}{A+1}\alpha + \frac{\beta(1-N)}{B} \geqslant 1,$$

because $A+1 \geqslant 2-M$.

Let us show that the inequality

$$\alpha\big(2-M\big)B + \beta\big(1-N\big)\big(A+1\big) \geqslant B\big(A+1\big)$$

holds. Let L_1 be the line in $\mathbb{R}^2$ given by the equation

$$x\big(2-M\big)B + y\big(1-N\big)(A+1) - B(A+1) = 0$$

and let L_2 be the line that is given by the equation

$$x\big(1-M\big) + Ay - A = 0,$$

where (x, y) are coordinates on $\mathbb{R}^2$. Then L_1 intersects the line $y = 0$ at the point

$$\left(\frac{A+1}{2-M}, 0\right)$$

and L_2 intersects the line $y = 0$ at the point $(A/(1-M), 0)$. But

$$\frac{A+1}{2-M} < \frac{A}{1-M},$$

which implies that $\alpha(2-M)B + \beta(1-N)(A+1) \geqslant B(A+1)$ if

$$A^2\beta_0\big(B+N-1\big) \geqslant \alpha_0\big(A+M-1\big),$$

where (α_0, β_0) is the intersection point of the lines L_1 and L_2. But

$$\big(\alpha_0, \beta_0\big) = \left(\frac{A(A+1)(B+N-1)}{\Delta}, \frac{B(A-1+M)}{\Delta}\right),$$

where $\Delta = 2AB - ABM - A + AM - 1 + M + NA - NAM + N - NM$. But

$$A^2\Big(B\big(A-1+M\big)\Big)\big(B+N-1\big) \geqslant \Big(A\big(A+1\big)\big(B+N-1\big)\Big)\big(A+M-1\big),$$

because $A(B-1) \geqslant 1$, which implies that $A^2\beta_0(B+N-1) \geqslant \alpha_0(A+M-1)$.

Finally, let us show that the inequality

$$\alpha\big(B+1-MB-N\big) + \beta\big(A+1-AN-M\big) \geqslant AB-1$$

holds. Let L_1' be the line in $\mathbb{R}^2$ given by the equation

$$x\big(B+1-MB-N\big) + y\beta\big(A+1-AN-M\big) - AB + 1 = 0$$

where (x, y) are coordinates on $\mathbb{R}^2$. Then L_1' intersects the line $y = 0$ at the point

$$\left(\frac{AB-1}{B+1-MB-N}, 0\right)$$

and L_2 intersects the line $y = 0$ at the point $(A/(1-M), 0)$. But

$$\frac{AB-1}{B+1-MB-N} < \frac{A}{1-M},$$

which implies that $\alpha(B+1-MB-N)+\beta(A+1-AN-M) \geqslant AB-1$ if

$$A^2\beta_1\big(B+N-1\big) \geqslant \alpha_1\big(A+M-1\big),$$

where (α_1, β_1) is the intersection point of the lines L_1' and L_2. Note that

$$(\alpha_1, \beta_1) = \left(\frac{A(AB-A-2+NA+M)}{\Delta'}, \ \frac{A+1-NA-M}{\Delta'}\right),$$

where $\Delta' = AB-1-ABM+AM+2M-NAM-M^2$.

To complete the proof, it is enough to show that the inequality

$$A^2\Big(A+1-NA-M\Big)(B+N-1) \geqslant \Big(A(AB-A-2+NA+M)\Big)(A+M-1)$$

holds. This inequality is equivalent to the inequality

$$\big(2-M\big)\big(A+M-1\big) \geqslant A\big(AN+2M-2\big)\big(B+N-1\big),$$

which is true, because $M \leqslant 1$ and $AN+2M-2 \leqslant 0$. $\square$

Let us prove prove Theorem 1.28 by reductio ad absurdum. Suppose that the inequalities

$$\mathrm{mult}_O\Big(D\cdot\Delta_1\Big) < M + Aa_1 - a_2 \text{ and } \mathrm{mult}_O\Big(D\cdot\Delta_2\Big) < N + Ba_2 - a_1$$

hold. Let us show that this assumption leads to a contradiction.

Lemma 3.2. The inequalities $a_1 > (1-M)/A$ and $a_2 > (1-N)/B$ hold.

Proof. It follows from Lemma 2.3 that

$$M + Aa_1 - a_2 > \mathrm{mult}_O\Big(D\cdot\Delta_1\Big) > 1 - a_2,$$

which implies that $a_1 > (1-M)/A$. Similarly, we see that $a_2 > (1-N)/B$. $\square$

Put $m_0 = \mathrm{mult}_O(D)$. Then m_0 is a positive rational number.

Remark 3.3. The inequalities $m_0 < M + Aa_1 - a_2$ and $m_0 < N + Ba_2 - a_1$ hold.

Lemma 3.4. The inequality $m_0 + a_1 + a_2 < 2$ holds.

Proof. We know that $m_0 + a_1 + a_2 < M + (A+1)a_1$ and $m_0 + a_1 + a_2 < N + (B+1)a_2$. Then

$$\big(m_0+a_1+a_2\big)\left(\frac{\alpha}{A+1}+\frac{\beta}{B+1}\right) < \alpha a_1 + \beta a_2 + \frac{\alpha M}{A+1} + \frac{\beta N}{B+1} \leqslant 1 + \frac{\alpha M}{A+1} + \frac{\beta N}{B+1},$$

which implies that $m_0 + a_1 + a_2 < 2$ by Lemma 3.1. $\square$

Let $\pi_1\colon S_1 \to S$ be the blow up of the point O, and let F_1 be the π_1-exceptional curve. Then

$$K_{S_1} + D^1 + a_1\Delta_1^1 + a_2\Delta_2^1 + \big(m_0+a_1+a_2-1\big)F_1 \sim_{\mathbb{Q}} \pi_1^*\Big(K_S + D + a_1\Delta_1 + a_2\Delta_2\Big),$$

where D^1, Δ_1^1, Δ_2^1 are proper transforms of the divisors D, Δ_1, Δ_2 via π_1, respectively. Then

$$\Big(S_1, \ D^1 + a_1\Delta_1^1 + a_2\Delta_2^1 + \big(m_0+a_1+a_2-1\big)F_1\Big)$$

is not Kawamata log terminal at some point $O_1 \in F_1$ (see Remark 2.4), where $m_0 + a_1 + a_2 \geqslant 1$.

Lemma 3.5. Either $O_1 = F_1 \cap \Delta_1^1$ or $O_1 = F_1 \cap \Delta_2^1$.

Proof. Suppose that $O_1 \notin \Delta_1^1 \cup \Delta_2^1$. Then $m_0 = D^1 \cdot F_1 > 1$ by Lemma 2.3. But

$$m_0\Bigg(\frac{\beta + B\alpha}{AB-1} + \frac{\alpha + A\beta}{AB-1}\Bigg) < \big(M + Aa_1 - a_2\big)\frac{\beta + B\alpha}{AB-1} + \big(N + Ba_2 - a_1\big)\frac{\alpha + A\beta}{AB-1},$$

because $m_0 < M + Aa_1 - a_2$ and $m_0 < N + Ba_2 - a_1$. On the other hand, we have

$$\big(M + Aa_1 - a_2\big)\frac{\beta + B\alpha}{AB-1} + \big(N + Ba_2 - a_1\big)\frac{\alpha + A\beta}{AB-1} \leqslant 1 + \frac{M\beta + MB\alpha + N\alpha + AN\beta}{AB-1},$$

because $\alpha a_1 + \beta a_2 \leqslant 1$ and $AB - 1 > 0$. But we already proved that $m_0 > 1$. Thus, we see that

$$\beta + B\alpha + \alpha + A\beta < AB - 1 + M\beta + MB\alpha + N\alpha + AN\beta,$$

which is impossible by Lemma 3.1. □

Lemma 3.6. The inequality $O_1 \neq F_1 \cap \Delta_1^1$ holds.

Proof. Suppose that $O_1 = F_1 \cap \Delta_1^1$. It follows from Lemma 2.3 that

$$M + Aa_1 - a_2 - m_0 > \mathrm{mult}_{O_1}\Big(D^1 \cdot \Delta_1^1\Big) > 1 - \big(m_0 + a_1 + a_2 - 1\big),$$

which implies that $a_1 > (2 - M)/(A + 1)$. Then

$$\frac{(2-M)\alpha}{A+1} + \frac{\beta(1-N)}{B} < \alpha a_1 + \beta a_2 \leqslant 1,$$

because $a_2 > (1 - N)/B$ by Lemma 3.2. Thus, we see that

$$\frac{(2-M)\alpha}{A+1} + \frac{\beta(1-N)}{B} < 1,$$

which is impossible by Lemma 3.1. □

Therefore, we see that $O_1 = F_1 \cap \Delta_2^1$. Then the log pair

$$\Big(S_1,\ D^1 + a_1\Delta_1^1 + a_2\Delta_2^1 + \big(m_0 + a_1 + a_2 - 1\big)F_1\Big)$$

is not Kawamata log terminal at the point O_1. We know that $1 > m_0 + a_1 + a_2 - 1 \geqslant 0$.

We have a blow up $\pi_1\colon S_1 \to S$. For any $n \in \mathbb{N}$, consider a sequence of blow ups

$$S_n \xrightarrow{\ \pi_n\ } S_{n-1} \xrightarrow{\ \pi_{n-1}\ } \cdots \xrightarrow{\ \pi_3\ } S_2 \xrightarrow{\ \pi_2\ } S_1 \xrightarrow{\ \pi_1\ } S$$

such that $\pi_{i+1}\colon S_{i+1} \to S_i$ is a blow up of the point $F_i \cap \Delta_2^i$ for every $i \in \{1, \ldots, n-1\}$, where

- we denote by F_i the exceptional curve of the morphism π_i,
- we denote by Δ_2^i the proper transform of the curve Δ_2 on the surface S_i.

For every $k \in \{1, \ldots, n\}$ and for every $i \in \{1, \ldots, k\}$, let D^k, Δ_1^k and F_i^k be the proper transforms on the surface S_k of the divisors D, Δ_1 and F_i, respectively. Then

$$K_{S_n} + D^n + a_1\Delta_1^n + a_2\Delta_2^n + \sum_{i=1}^{n}\Bigg(a_1 + ia_2 - i + \sum_{j=0}^{i-1} m_j\Bigg)F_i^n \sim_{\mathbb{Q}} \pi^*\Big(K_S + D + a_1\Delta_1 + a_2\Delta_2\Big),$$

where $\pi = \pi_n \circ \cdots \circ \pi_2 \circ \pi_1$ and $m_i = \mathrm{mult}_{O_i}(D^i)$ for every $i \in \{1, \ldots, n\}$. Then the log pair

$$\Bigg(S_n,\ D^n + a_1\Delta_1^n + a_2\Delta_2^n + \sum_{i=1}^{n}\Bigg(a_1 + ia_2 - i + \sum_{j=0}^{i-1} m_j\Bigg)F_i^n\Bigg) \tag{3.7}$$

is not Kawamata log terminal at some point of the set $F_1^n \cup F_2^n \cup \cdots \cup F_n^n$ (see Remark 2.4).

Put $O_k = F_k \cap \Delta_2^k$ for every $k \in \{1, \ldots, n\}$.

Lemma 3.8. For every $i \in \{1, \ldots, n\}$, we have

$$1 > a_1 + ia_2 - i + \sum_{j=0}^{i-1} m_j \geqslant 0,$$

and (3.7) is Kawamata log terminal at every point of the set $(F_1^n \cup F_2^n \cup \cdots \cup F_n^n) \setminus O_n$.

Since $\mathrm{mult}_O(D \cdot \Delta_2) < N + Ba_2 - a_1$ by assumption, it follows from Lemma 3.8 that

$$N + Ba_2 - a_1 > \mathrm{mult}_O\Big(D \cdot \Delta_2\Big) \geqslant \sum_{i=0}^{n-1} m_i \geqslant (n-1)(1-a_2) - a_1,$$

which implies that $n \leqslant (N+Ba_2)/(1-a_2)$. On the other hand, the assertion of Lemma 3.8 holds for arbitrary $n \in \mathbb{N}$. So, taking any $n > (N + Ba_2)/(1 - a_2)$, we obtain a contradiction.

We see that to prove Theorem 1.28, it is enough to prove Lemma 3.8.

Let us prove Lemma 3.8 by induction on $n \in \mathbb{N}$. The case $n = 1$ is already done.

We may assume that $n \geqslant 2$. For every $k \in \{1, \ldots, n-1\}$, we may assume that

$$1 > a_1 + ka_2 - k + \sum_{j=0}^{k-1} m_j \geqslant 0,$$

the singularities of the log pair

$$\left(S_k,\ D^k + a_1\Delta_1^k + a_2\Delta_2^k + \sum_{i=1}^{k}\left(a_1 + ka_2 - k + \sum_{j=0}^{i-1} m_j\right)F_i^k\right)$$

are Kawamata log terminal along $(F_1^k \cup F_2^k \cup \cdots \cup F_k^k) \setminus O_k$ and not Kawamata log terminal at O_k.

Lemma 3.9. The inequality $a_2 > (n-N)/(B+n-1)$ holds.

Proof. The singularities of the log pair

$$\left(S_{n-1},\ D^{n-1} + a_2\Delta_2^{n-1} + \left(a_1 + \big(n-1\big)a_2 - \big(n-1\big) + \sum_{j=0}^{n-2} m_j\right)F_{n-1}^{n-1}\right)$$

are not Kawamata log terminal at the point O_{n-1}. Then it follows from Lemma 2.3 that

$$N + Ba_2 - a_1 - \sum_{j=0}^{n-2} m_j > \mathrm{mult}_{O_{n-1}}\Big(D^{n-1} \cdot \Delta_2^{n-1}\Big) > 1 - \left(a_1 + \big(n-1\big)a_2 - \big(n-1\big) + \sum_{j=0}^{n-2} m_j\right),$$

which implies that $a_2 > (n-N)/(B+n-1)$. □

Lemma 3.10. The inequalities $1 > a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \geqslant 0$ hold.

Proof. The inequality $a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \geqslant 0$ follows from the fact that the log pair

$$\left(S_{n-1},\ D^{n-1} + a_2\Delta_2^{n-1} + \left(a_1 + \big(n-1\big)a_2 - \big(n-1\big) + \sum_{j=0}^{n-2} m_j\right)F_{n-1}^{n-1}\right)$$

is not Kawamata log terminal at the point O_{n-1}.

Suppose that $a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \geqslant 1$. Let us derive a contradiction.

It follows from Remark 3.3 that $m_0 + a_2 \leqslant M + Aa_1$. Then

$$a_1 + nM + nAa_1 - n \geqslant a_1 + na_2 - n + nm_0 \geqslant a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j \geqslant 1,$$

which implies that $a_1 \geqslant (n+1-Mn)/(nA+1)$. But $a_2 > (n-N)/(B+n-1)$ by Lemma 3.9. Then

$$\left(\frac{\alpha(1-M)}{A}+\beta\right)+\alpha\frac{A-1+M}{A(An+1)}+\beta\frac{1-B-N}{B+n-1}=\alpha\frac{n+1-Mn}{nA+1}+\beta\frac{n-N}{B+n-1}<\alpha a_1+\beta a_2\leqslant 1,$$

where $\alpha(1-M)/A+\beta \geqslant 1$ by assumption. Therefore, we see that

$$\alpha\frac{A+M-1}{A(An+1)}<\beta\frac{B+N-1}{B+n-1},$$

where $n \geqslant 2$. But $A+M>1$ and $B+N>1$ by Lemma 3.2, since $a_1<1$ and $a_2<1$. Then

$$\frac{A(An+1)}{\alpha(A+M-1)}>\frac{B+n-1}{\beta(B+N-1)},$$

but $A^2(B+N-1)\beta \leqslant \alpha(A+M-1)$ by assumption. Then

$$\frac{A}{\alpha(A+M-1)}-\frac{B-1}{\beta(B+N-1)}\geqslant\left(\frac{A^2}{\alpha(A+M-1)}-\frac{1}{\beta(B+N-1)}\right)n+\frac{A}{\alpha(A+M-1)}-\frac{B-1}{\beta(B+N-1)}>0,$$

which implies that $\beta A(B+N-1)>\alpha(B-1)(A+M-1)$. Then

$$\frac{\alpha(A+M-1)}{A}\geqslant\beta A\big(B+N-1\big)>\alpha\big(B-1\big)\big(A+M-1\big),$$

because $A^2(B+N-1)\beta\leqslant\alpha(A+M-1)$ by assumption. Then we have $\alpha\neq 0$ and $A(B-1)<1$, which is impossible, because $A(B-1)\geqslant 1$ by assumption. □

Lemma 3.11. The log pair (3.7) is Kawamata log terminal at every point of the set

$$F_n\setminus\left(\Big(F_n\cap F_{n-1}^n\Big)\bigcup\Big(F_n\cap\Delta_2^n\Big)\right).$$

Proof. Suppose that there is a point $Q\in F_n$ such that

$$F_n\cap F_{n-1}^n\neq Q\neq F_n\cap\Delta_2^n,$$

but (3.7) is not Kawamata log terminal at the point Q. Then the log pair

$$\left(S_n,\ D^n+\left(a_1+na_2-n+\sum_{j=0}^{n-1}m_j\right)F_n\right)$$

is not Kawamata log terminal at the point Q as well. Then

$$m_0\geqslant m_{n-1}=D^n\cdot F_n>1$$

by Lemma 2.3, because $a_1+na_2-n+\sum_{j=0}^{n-1}m_j<1$ by Lemma 3.10. Then

$$m_0\left(\frac{\beta+B\alpha}{AB-1}+\frac{\alpha+A\beta}{AB-1}\right)<\big(M+Aa_1-a_2\big)\frac{\beta+B\alpha}{AB-1}+\big(N+Ba_2-a_1\big)\frac{\alpha+A\beta}{AB-1},$$

because $m_0<M+Aa_1-a_2$ and $m_0<N+Ba_2-a_1$ by Remark 3.3. We have

$$\big(M+Aa_1-a_2\big)\frac{\beta+B\alpha}{AB-1}+\big(N+Ba_2-a_1\big)\frac{\alpha+A\beta}{AB-1}\leqslant 1+\frac{M\beta+MB\alpha+N\alpha+AN\beta}{AB-1},$$

because $\alpha a_1+\beta a_2\leqslant 1$ and $AB-1>0$. But $m_0>1$. Thus, we see that

$$\beta+B\alpha+\alpha+A\beta<AB-1+M\beta+MB\alpha+N\alpha+AN\beta,$$

which contradicts our initial assumptions. □

Lemma 3.12. The log pair (3.7) is Kawamata log terminal at the point $F_n\cap F_{n-1}^n$.

Proof. Suppose that (3.7) is not Kawamata log terminal at $F_n \cap F_{n-1}^n$. Then the log pair

$$\left(S_n,\ D^n + \left(a_1 + (n-1)a_2 - (n-1) + \sum_{j=0}^{n-2} m_j\right)F_{n-1}^n + \left(a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j\right)F_n\right)$$

is not Kawamata log terminal at the point $F_n \cap F_{n-1}^n$ as well. Then

$$m_{n-2} - m_{n-1} = D^n \cdot F_{n-2} > 1 - \left(a_1 + na_2 - n + \sum_{j=0}^{n-1} m_j\right)$$

by Lemma 2.3, because $a_1 + (n-1)a_2 - (n-1) + \sum_{j=0}^{n-2} m_j < 1$. Note that

$$M + Aa_1 - a_2 - m_0 > \mathrm{mult}_O\Big(D \cdot \Delta_1\Big) - m_0 \geqslant \mathrm{mult}_O\big(D\big)\mathrm{mult}_O\big(\Delta_1\big) - m_0 = 0,$$

which implies that $m_0 + a_2 < Aa_1 + M$. Then

$$nM + nAa_1 - na_2 > nm_0 \geqslant m_{n-2} - m_{n-1} + \sum_{j=0}^{n-1} m_j > n + 1 - a_1 - na_2,$$

which gives $a_1 > (n + 1 - nM)/(An + 1)$.

Now arguing as in the proof of Lemma 3.10, we obtain a contradiction. □

The assertion of Lemma 3.8 is proved. The assertion of Theorem 1.28 is proved.

4. One cyclic singular point

Let X be a sextic surface in $\mathbb{P}(1, 1, 2, 3)$ with canonical singularities such that $|\mathrm{Sing}(X)| = 1$, let $\omega\colon X \to \mathbb{P}(1, 1, 2)$ be the natural double cover, let R be its ramification curve in $\mathbb{P}(1, 1, 2)$, and suppose that $\mathrm{Sing}(X)$ consists of one singular point of type $\mathbb{A}_m$, where $m \in \{1, \ldots, 8\}$.

Theorem 4.1. The following equality holds:

$$\mathrm{lct}\big(X\big) = \begin{cases} \mathrm{lct}_3\big(X\big) = 1/2 \text{ if } m = 8,\\ \mathrm{lct}_2\big(X\big) = 1/2 \text{ if } m = 7 \text{ and } R \text{ is reducible},\\ \mathrm{lct}_3\big(X\big) = 3/5 \text{ if } m = 7 \text{ and } R \text{ is irreducible},\\ \mathrm{lct}_2\big(X\big) = 2/3 \text{ if } m = 6,\\ \mathrm{lct}_2\big(X\big) = 2/3 \text{ if } m = 5,\\ \mathrm{lct}_2\big(X\big) = 4/5 \text{ if } m = 4,\\ \mathrm{lct}_1\big(X\big) \text{ in the remaining cases,} \end{cases}$$

and if $\mathrm{lct}(X) = 2/3$, then there is a unique effective $\mathbb{Q}$-divisor D on X such that $D \sim_{\mathbb{Q}} -K_X$ and

$$\mathrm{c}\big(X, D\big) = \mathrm{lct}\big(X\big) = \frac{2}{3}.$$

By Theorem 1.5, Corollary 1.12 and Remark 1.8, we obtain the following two corollaries.

Corollary 4.2. If $m \leqslant 6$, then $\mathrm{lct}_{n,2}(X) > 2/3$ for every $n \in \mathbb{N}$.

Corollary 4.3. If $m \leqslant 6$, then X is Kähler–Enstein.

In the rest of this section we will prove Theorem 4.1.

Let D be an arbitrary effective $\mathbb{Q}$-divisor on the surface X such that

$$D \sim_{\mathbb{Q}} -K_X,$$

and put $\mu = \mathrm{c}(X, D)$. To prove Theorem 4.1, it is enough to show that

$$\mu \geqslant \begin{cases} \mathrm{lct}_3\big(X\big) = 1/2 \text{ if } m = 8, \\ \mathrm{lct}_2\big(X\big) = 1/2 \text{ if } m = 7 \text{ and } R \text{ is reducible,} \\ \mathrm{lct}_3\big(X\big) = 3/5 \text{ if } m = 7 \text{ and } R \text{ is irreducible,} \\ \mathrm{lct}_2\big(X\big) = 2/3 \text{ if } m = 6, \\ \mathrm{lct}_2\big(X\big) = 2/3 \text{ if } m = 5, \\ \mathrm{lct}_2\big(X\big) = 4/5 \text{ if } m = 4, \\ \mathrm{lct}_1\big(X\big) \text{ in the remaining cases,} \end{cases}$$

and if $\mu = \mathrm{lct}(X) = 2/3$, then D is uniquely defined. Note that $\mathrm{lct}_1(X) \geqslant 5/6$ if $m \geqslant 3$ (see [30]).

Let us prove Theorem 4.1. By Lemma 2.6, we may assume that $m \geqslant 3$ and $\mu < \mathrm{lct}_1(X)$. Then

$$\mathrm{LCS}\big(X, \mu D\big) = \mathrm{Sing}\big(X\big)$$

by Lemma 2.6. Put $P = \mathrm{Sing}(X)$.

Let $\pi\colon \bar{X} \to X$ be a minimal resolution, let $E_1, E_2, \ldots, E_m$ be π-exceptional curves such that

$$E_i \cdot E_j \neq 0 \iff \big|i - j\big| \leqslant 1,$$

let C be the curve in $|-K_X|$ such that $P \in C$, and let $\bar{C}$ be it proper transform on $\bar{X}$. Then

$$\bar{C} \sim_{\mathbb{Q}} \pi^*\big(C\big) - \sum_{i=1}^{m} E_i,$$

and the curve C is irreducible. We may assume that $D \neq C$, because $\mu \geqslant \mathrm{lct}_1\big(X\big)$ if $D = C$.

By Remark 2.1, we may assume that $C \not\subset \mathrm{Supp}(D)$.

Let $\bar{D}$ be the proper transform of the divisor D on the surface $\bar{X}$. Then

$$\bar{D} \sim_{\mathbb{Q}} \pi^*\big(D\big) - \sum_{i=1}^{m} a_i E_i,$$

where a_i is a non-negative rational number. Then the log pair

$$\Big(\bar{X}, \mu\bar{D} + \sum_{i=1}^{m} \mu a_i E_i\Big) \tag{4.4}$$

is not Kawamata log terminal (by Remark 2.4). On the other hand, we have

$$\bar{D}\cdot E_1 = 2a_1 - a_2,\ \bar{D}\cdot E_2 = 2a_2 - a_1 - a_3,\ \cdots,\ \bar{D}\cdot E_{m-1} = 2a_{m-1} - a_{m-2} - a_m,\ \bar{D}\cdot E_m = 2a_m - a_{m-1},$$

where all intersections $\bar{D}\cdot E_1, \bar{D}\cdot E_2, \ldots, \bar{D}\cdot E_m$ are non-negative. Moreover, we have

$$\bar{D}\cdot\bar{C} = 1 - a_1 - a_m,$$

where the intersection $\bar{D}\cdot\bar{C}$ is non-negative, since $C\not\subset \mathrm{Supp}(D)$ by assumption. Hence, we have

$$
\begin{cases}
a_1\geqslant \dfrac{a_2}{2},\\
a_2\geqslant \dfrac{a_1+a_3}{2},\\
a_3\geqslant \dfrac{a_2+a_4}{2},\\
\cdots\\
a_{m-1}\geqslant \dfrac{a_{m-2}+a_m}{2},\\
a_m\geqslant \dfrac{a_{m-1}}{2},\\
1\geqslant a_1+a_m.
\end{cases}
\tag{4.5}
$$

It should be pointed out that at least one inequality in (4.5) must be strict, since $\bar{D}\cdot E_i>0$ for at least one $i\in\{1,\ldots,m\}$, because $P\in\mathrm{Supp}(D)$. Then $a_i>0$ for some $i\in\{1,\ldots,m\}$.

Note that $a_1\geqslant a_2/2$ by (4.5). Similarly, it follows from (4.5) that

$$a_2\geqslant \frac{a_1+a_3}{2}\geqslant \frac{a_1}{4}+\frac{a_3}{4},$$

which implies that $a_2\geqslant 2a_3/3$. Arguing in the same way, we see that

$$a_k\geqslant \frac{k}{k+1}a_{k+1}$$

for every $k\in\{1,\ldots,m-1\}$ (use (4.5) and induction on k). Using symmetry, we see that

$$a_{k+1}\geqslant \frac{m-k}{m-k+1}a_k$$

for every $k\in\{1,\ldots,m-1\}$. In particular, the inequality $a_k>0$ holds for every $k\in\{1,\ldots,m\}$, since we already know that $a_i>0$ for some $i\in\{1,\ldots,m\}$.

Lemma 4.6. Suppose that $\mu a_i<1$ for every $i\in\{1,\ldots,m\}$. Then

- there exists a point

$$Q\in\Big\{E_1\cap E_2, E_2\cap E_3,\ldots,E_{m-1}\cap E_m\Big\}$$

  such that the log pair (4.4) is not Kawamata log terminal at Q,
- the log pair (4.4) is Kawamata log terminal outside of the point Q,
- if $\mu<(m+1)/(2m-2)$, then $Q\neq E_1\cap E_2$ and $Q\neq E_{m-1}\cap E_m$.

Proof. It follows from Remark 2.4 and Theorem 2.2 that there is a point $Q\in\cup_{i=1}^m E_i$ such that the log pair (4.4) is not Kawamata log terminal at Q and is Kawamata log terminal elsewhere.

Suppose that $Q\in E_1$ and $Q\notin E_2$. Then

$$2a_1-a_2=\bar{D}\cdot E_i>1$$

by Lemma 2.3. Taking (4.5) into account, we get

$$\begin{cases} a_1 > \dfrac{1}{2} + \dfrac{a_2}{2}, \\ a_2 \geqslant \dfrac{a_1 + a_3}{2}, \\ a_3 \geqslant \dfrac{a_2 + a_4}{2}, \\ \cdots \\ a_{m-1} \geqslant \dfrac{a_{m-2} + a_m}{2}, \\ a_m \geqslant \dfrac{a_{m-1}}{2}, \end{cases}$$

and adding all these inequalities together we get

$$\sum_{i=1}^{m} a_i > \frac{1}{2} + \frac{a_1}{2} + \sum_{i=2}^{m-1} a_i + \frac{a_m}{2},$$

which implies that $a_1 + a_m > 1$. However, the later is impossible, since $a_1 + a_m \leqslant 1$ by (4.5).

We see that if $Q \in E_1$, then $Q = E_1 \cap E_2$. Similarly, we see that $Q = E_{m-1} \cap E_m$ if $Q \in E_m$.

Suppose that $Q \in E_i$ and $Q \notin E_j$ for every $j \neq i$. Then $i \neq 1$ and $i \neq m$. We have

$$2a_i - a_{i-1} - a_{i+1} = \bar{D} \cdot E_i > 1$$

by Lemma 2.3. Taking (4.5) into account, we get

$$\begin{cases} a_1 > \dfrac{a_2}{2}, \\ a_2 \geqslant \dfrac{a_1 + a_3}{2}, \\ a_3 \geqslant \dfrac{a_2 + a_4}{2}, \\ \cdots \\ a_i \geqslant \dfrac{1}{2} + \dfrac{a_{i-1} + a_{i+1}}{2}, \\ \cdots \\ a_{m-1} \geqslant \dfrac{a_{m-2} + a_m}{2}, \\ a_m \geqslant \dfrac{a_{m-1}}{2}, \end{cases}$$

and adding all these inequalities together we get

$$\sum_{i=1}^{m} a_i > \frac{1}{2} + \frac{a_1}{2} + \sum_{i=2}^{m-1} a_i + \frac{a_m}{2},$$

which implies that $a_1 + a_m > 1$. However, the later is impossible, since $a_1 + a_m \leqslant 1$ by (4.5).

Thus, we see that there is $k \in \{1, \ldots, m-1\}$ such that $Q = E_k \cap E_{k+1}$.

Suppose that $\mu < (m+1)/(2m-2)$. Let us show that $k \neq 1$ and $k \neq m-1$.

Due to symmetry, it is enough to show that $k \neq 1$. Recall that $m \geqslant 3$.

Suppose that $k = 1$. Then $Q = E_1 \cap E_2$. Take $\bar{\mu} \in \mathbb{Q}$ such that $(m+1)/(2m-2) > \bar{\mu} > \mu$ and

$$\left(\bar{X}, \mu\bar{D} + \bar{\mu}a_1E_1 + \bar{\mu}a_2E_2\right)$$

is not Kawamata log terminal at Q and is Kawamata log terminal outside of the point Q. Then

$$\frac{2m-2}{m+1}\bar{\mu}a_1+\frac{2}{m+1}\bar{\mu}a_2<a_1+\frac{1}{m-1}a_2\leqslant 1,$$

by (4.5), since $a_1\neq 0$ and $a_2\neq 0$. On the other hand, we have

$$\mathrm{mult}_Q\Big(\mu\bar{D}\cdot E_1\Big)\leqslant \mu\bar{D}\cdot E_1=\mu\Big(2a_1-a_2\Big)<\bar{\mu}\Big(2a_1-a_2\Big),$$

since $\mu<\bar{\mu}$. Therefore, it follows from Corollary 1.29 that

$$\mu\Big(2a_2-a_1-a_3\Big)=\mu\bar{D}\cdot E_2\geqslant \mathrm{mult}_Q\Big(\mu\bar{D}\cdot E_2\Big)\geqslant\frac{m}{m-1}\bar{\mu}a_2-\bar{\mu}a_1,$$

which implies that $a_2(m-2)>a_3(m-1)$, since $\mu<\bar{\mu}$. But we proved earlier that

$$a_3\geqslant\frac{m-2}{m-1}a_2,$$

which is impossible, since $a_2(m-2)>a_3(m-1)$. Thus, we see that $k\neq 1$. □

If $m=3$, then it follows from (4.5) that $a_1\leqslant 3/4$, $a_2\leqslant 1$, $a_3\leqslant 3/4$.

Corollary 4.7. If $m=3$, then $\mu\geqslant \mathrm{lct}_1(X)\geqslant 5/6$.

Lemma 4.8. Suppose that $m=4$. Then $\mu\geqslant\mathrm{lct}_2(X)=4/5$.

Proof. There is a unique smooth irreducible curve $\bar{Z}\subset\bar{X}$ such that

$$\bar{Z}\sim\pi^*\big(-2K_X\big)-E_1-2E_2-2E_3-E_4$$

and $E_2\cap E_3\in\bar{Z}$ (cf. the proof of Lemma 6.9). Put $Z=\pi(\bar{Z})$. Then

$$\mathrm{lct}_2\big(X\big)\leqslant \mathrm{c}\Big(X,\frac{1}{2}Z\Big)=\frac{4}{5}.$$

To complete the proof, it is enough to show that $\mu\geqslant 4/5$. Suppose that $\mu<4/5$.

By Remark 2.1, we may assume that $Z\not\subset\mathrm{Supp}(D)$, because Z is irreducible.

It follows from (4.5) that $a_1\leqslant 4/5$, $a_2\leqslant 6/5$, $a_3\leqslant 6/5$, $a_4\leqslant 4/5$.

Put $Q=E_2\cap E_3$. Then it follows from Lemma 4.6 that (4.4) is not Kawamata log terminal at the point Q and is Kawamata log terminal outside of the point Q. Then

$$2a_2-\frac{1}{2}a_2-a_3\geqslant 2a_2-a_1-a_3=\bar{D}\cdot E_2\geqslant\mathrm{mult}_Q\Big(\bar{D}\cdot E_2\Big)>\frac{5}{4}-a_3,$$

by Lemma 2.3. Similarly, we see that

$$2a_3-a_2-a_4=\bar{D}\cdot E_3\geqslant\mathrm{mult}_Q\Big(\bar{D}\cdot E_3\Big)>\frac{5}{4}-a_2,$$

which implies that $a_2>5/6$ and $a_3>5/6$.

Let $\xi\colon\tilde{X}\to\bar{X}$ be a blow up of the point Q, let E be the exceptional curve of the blow up ξ, and let $\tilde{D}$ be the proper transform of the divisor $\bar{D}$ on the surface $\tilde{X}$. Put $\delta=\mathrm{mult}_Q(\bar{D})$.

Let $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$, $\tilde{E}_4$ be the proper transforms on $\tilde{X}$ of E_1, E_2, E_3, E_4, respectively. Then

$$\Big(\tilde{X},\mu\tilde{D}+\mu a_2\tilde{E}_2+\mu a_3\tilde{E}_3+\big(\mu a_2+\mu a_3+\mu\delta-1\big)E\Big)\tag{4.9}$$

is not Kawamata log terminal at some point $O\in E$.

Let $\tilde{Z}$ be the proper transform on $\tilde{X}$ of the curve $\bar{Z}$. Then

$$0\leqslant\tilde{Z}\cdot\tilde{D}=2-a_2-a_3-\mathrm{mult}_Q\big(\bar{D}\big)=2-a_2-a_3-\delta,$$

which implies that $\delta + a_2 + a_3 \leqslant 2$. We have $\mu a_2 + \mu a_3 + \mu\delta - 1 \leqslant 2\mu - 1 \leqslant 3/5$, which implies that (4.9) is Kawamata log terminal outside of the point O by Theorem 2.2. We have

$$\begin{cases} 2a_3 - a_2 - a_4 - \delta = \tilde{E}_3 \cdot \tilde{D} \geqslant 0, \\ 2a_2 - a_1 - a_3 - \delta = \tilde{E}_2 \cdot \tilde{D} \geqslant 0, \end{cases}$$

which implies that $\delta \leqslant 1$. If $O \notin \tilde{E}_2 \cup \tilde{E}_3$, then

$$1 \geqslant \delta = \tilde{D} \cdot E \geqslant \mathrm{mult}_O\Big(\tilde{D} \cdot E\Big) > \frac{5}{4}$$

by Lemma 2.3. Thus, we see that either $O = \tilde{E}_2 \cap E$ or $O = \tilde{E}_3 \cap E$.

Without loss of generality, we may assume that $O = \tilde{E}_2 \cap E$. By Lemma 2.3, one has

$$\frac{5}{4} - a_2 > \frac{7}{6} - a_2 = 2 - \frac{5}{6} - a_2 > 2 - a_2 - a_3 \geqslant \delta = \tilde{D} \cdot E \geqslant \mathrm{mult}_O\Big(\tilde{D} \cdot E\Big) > \frac{5}{4} - a_2,$$

since $\delta + a_2 + a_3 \leqslant 2$ and $a_3 > 5/6$. The obtained contradiction concludes the proof. □

Let τ be a biregular involution of the surface $\bar{X}$ that is induced by the double cover ω.

Lemma 4.10. Suppose that $m = 5$. Then there exists a unique curve $Z \in |-2K_X|$ such that

$$\mathrm{c}\left(X, \frac{1}{2}Z\right) = \mathrm{lct}_2\big(X\big) = \frac{2}{3},$$

and either $D = Z/2$ or $\mu > 2/3$.

Proof. Let $\alpha\colon \bar{X} \to \check{X}$ be a contraction of the curves $\bar{C}$, E_5, E_4, E_3. Then

$$\alpha\big(E_1\big) \cdot \alpha\big(E_1\big) = \alpha\big(E_2\big) \cdot \alpha\big(E_2\big) = -1,$$

and $\check{X}$ is a smooth del Pezzo surface such that $K^2_{\check{X}} = 5$, which implies that there is a smooth irreducible rational curve $\check{L}_2$ on the surface $\check{X}$ such that $\check{L}_2 \cdot \alpha(E_2) = 1$ and $\check{L}_2 \cdot \check{L}_2 = -1$.

Let $\bar{L}_2$ be the proper transform of the curve $\check{L}_2$ on the surface $\bar{X}$. Then $\bar{L}_2 \cdot \bar{L}_2 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_2 = E_2 \cdot \bar{L}_2 = 1,$$

which implies that $E_1 \cdot \bar{L}_2 = E_3 \cdot \bar{L}_2 = E_4 \cdot \bar{L}_2 = E_5 \cdot \bar{L}_2 = \bar{C} \cdot \bar{L}_2 = 0$.

Let $\beta\colon \bar{X} \to \check{X}$ be a contraction of the curves $\bar{L}_2$, $\bar{C}$, E_5, E_4. Then

$$\beta\big(E_2\big) \cdot \beta\big(E_2\big) = \beta\big(E_3\big) \cdot \beta\big(E_3\big) = -1,$$

and $\check{X}$ is a smooth del Pezzo surface such that $K^2_{\check{X}} = 5$, which implies that there is an irreducible smooth curve $\check{L}_3 \subset \check{X}$ such that $\check{L}_3 \cdot \beta(E_3) = 1$ and $\check{L}_3 \cdot \check{L}_3 = -1$ (cf. the proof of Lemma 6.8).

Let $\bar{L}_3$ be the proper transform of the curve $\check{L}_3$ on the surface $\bar{X}$. Then $\bar{L}_3 \cdot \bar{L}_3 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_3 = E_3 \cdot \bar{L}_3 = 1,$$

which implies that $E_1 \cdot \bar{L}_3 = E_2 \cdot \bar{L}_3 = E_4 \cdot \bar{L}_3 = E_5 \cdot \bar{L}_3 = \bar{C} \cdot \bar{L}_3 = 0$.

If $\tau(\bar{L}_3) = \bar{L}_3$, then $2\pi(\bar{L}_3) \sim -2K_X$, but $\pi(\bar{L}_3)$ is not a Cartier divisor.

Put $Z = \pi(\bar{L}_3 + \tau(\bar{L}_3))$. Then $Z \sim -2K_X$ and $\mathrm{c}(X, Z) = 1/3$. We see that $\mathrm{lct}_2(X) \leqslant 2/3$.

Suppose that $D \neq Z/2$. To complete the proof, it is enough to show that $\mu > 2/3$.

Suppose that $\mu \leqslant 2/3$. Let us derive a contradiction. It follows from (4.5) that

$$a_1 \leqslant \frac{5}{6},\ a_2 \leqslant \frac{4}{3},\ a_3 \leqslant \frac{3}{2},\ a_4 \leqslant \frac{4}{3},\ a_5 \leqslant \frac{5}{6}.$$

By Remark 2.1, without loss of generality we may assume that $\pi(\bar{L}_3) \not\subset \mathrm{Supp}(D)$. Then

$$1 - a_3 = \bar{L}_3 \cdot \bar{D} \geqslant 0,$$

which implies that $a_3 \leqslant 1$.

Put $Q = E_2 \cap E_3$. By Lemma 4.6, we may assume that (4.4) is not Kawamata log terminal at the point Q and is Kawamata log terminal outside of the point Q. Then

$$2a_3 - a_2 - a_4 = \bar{D} \cdot E_3 \geqslant \mathrm{mult}_Q\Big(\bar{D} \cdot E_3\Big) \geqslant \frac{1}{\mu} - a_2 > \frac{3}{2} - a_2$$

by Lemma 2.3, which implies that $a_3 > 9/8$ by (4.5). But $a_3 \leqslant 1$. ☐

Lemma 4.11. Suppose that $m = 6$. Then there exists a unique curve $Z \in |-2K_X|$ such that

$$\mathrm{c}\left(X, \frac{1}{2}Z\right) = \mathrm{lct}_2\big(X\big) = \frac{2}{3}$$

and either $D = Z/2$ or $\mu > 2/3$.

Proof. Let $\alpha\colon \bar{X} \to \check{X}$ be a contraction of the curves $\bar{C}$, E_6, E_5, E_4 and E_3. Then

$$\alpha\big(E_1\big) \cdot \alpha\big(E_1\big) = \alpha\big(E_2\big) \cdot \alpha\big(E_2\big) = -1,$$

and $\check{X}$ is a smooth del Pezzo surface such that $K_{\check{X}}^2 = 6$, which implies that there is a smooth irreducible rational curve $\check{L}_2$ on the surface $\check{X}$ such that $\check{L}_2 \cdot \alpha(E_2) = 1$ and $\check{L}_2 \cdot \check{L}_2 = -1$.

Let $\bar{L}_2$ be the proper transform of the curve $\check{L}_2$ on the surface $\bar{X}$. Then $\bar{L}_2 \cdot \bar{L}_2 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_2 = E_2 \cdot \bar{L}_2 = 1,$$

which implies that $E_1 \cdot \bar{L}_2 = E_3 \cdot \bar{L}_2 = E_4 \cdot \bar{L}_2 = E_5 \cdot \bar{L}_2 = E_6 \cdot \bar{L}_2 = \bar{C} \cdot \bar{L}_2 = 0$.

Let $\beta\colon \bar{X} \to \check{X}$ be a contraction of the curves $\bar{L}_2$, $\bar{C}$, E_6, E_5 and E_4. Then

$$\beta\big(E_2\big) \cdot \beta\big(E_2\big) = \beta\big(E_3\big) \cdot \beta\big(E_3\big) = -1,$$

and $\check{X}$ is a smooth del Pezzo surface such that $K_{\check{X}}^2 = 6$, which implies that there are irreducible smooth rational curves $\check{L}_3$ and $\check{L}_2'$ on the surface $\check{X}$ such that

$$\check{L}_3 \cdot \beta\big(E_3\big) = \check{L}_2' \cdot \beta\big(E_2\big) = 1$$

and $\check{L}_3 \cdot \check{L}_3 = \check{L}_2' \cdot \check{L}_2' = -1$. Let $\bar{L}_3$ and $\bar{L}_2'$ be the proper transforms of the curves $\check{L}_3$ and $\check{L}_2'$ on the surface $\bar{X}$, respectively. Then $\bar{L}_3 \cdot \bar{L}_3 = \bar{L}_2' \cdot \bar{L}_2' = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_3 = -K_{\bar{X}} \cdot \bar{L}_2' = E_3 \cdot \bar{L}_3 = E_2 \cdot \bar{L}_2' = 1,$$

which implies that $\bar{C} \cdot \bar{L}_3 = \bar{C} \cdot \bar{L}_2' = 0$, and $E_i \cdot \bar{L}_3 = E_j \cdot \bar{L}_2' = 0$ for every $i \neq 3$ and $j \neq 2$,

Put $\bar{L}_4 = \tau(\bar{L}_3)$, $\bar{L}_5 = \tau(\bar{L}_2)$, $\bar{L}_5' = \tau(\bar{L}_2')$. Then $\bar{C} \cdot \bar{L}_4 = \bar{C} \cdot \bar{L}_5 = \bar{C} \cdot \bar{L}_5' = 0$ and

$$-K_{\bar{X}} \cdot \bar{L}_4 = -K_{\bar{X}} \cdot \bar{L}_5 = -K_{\bar{X}} \cdot \bar{L}_5' = E_4 \cdot \bar{L}_4 = E_5 \cdot \bar{L}_5 = E_5 \cdot \bar{L}_5' = 1,$$

which implies that $E_i \cdot \bar{L}_5 = E_i \cdot \bar{L}_5' = E_j \cdot \bar{L}_4 = 0$ for every $i \neq 5$ and $j \neq 4$.

Put $L_3 = \pi(\bar{L}_3)$, $L_4 = \pi(\bar{L}_4)$, $L_2 = \pi(\bar{L}_2)$, $L_2' = \pi(\bar{L}_2')$, $L_5 = \pi(\bar{L}_5)$, $L_5' = \pi(\bar{L}_5')$. Then

$$L_3 + L_4 \sim L_2 + L_5 \sim L_2' + L_5' \sim -2K_X,$$

and $\mathrm{c}(X, L_3 + L_4) = 1/3$, which implies that $\mathrm{lct}_2(X) \leqslant 2/3$.

Note that $\mathrm{c}(X, L_2 + L_5) = \mathrm{c}(X, L_2' + L_5') = 1/2$.

Suppose that $D \neq (L_3 + L_4)/2$. To complete the proof, it is enough to show that $\mu > 2/3$.

Suppose that $\mu \leqslant 2/3$. Let us derive a contradiction.

It follows from (4.5) that $a_1 \leqslant 6/7$, $a_2 \leqslant 10/7$, $a_3 \leqslant 12/7$, $a_4 \leqslant 12/7$, $a_5 \leqslant 10/7$, $a_6 \leqslant 6/7$.

By Remark 2.1, without loss of generality we may assume that $\bar{L}_4 \not\subset \mathrm{Supp}(D)$. Then

$$1 - a_4 = \bar{L}_3 \cdot \bar{D} \geqslant 0,$$

which gives us $a_4 \leqslant 1$. Similarly, we may assume that either $\bar{L}_2 \not\subset \mathrm{Supp}(D)$ or $\bar{L}_5 \not\subset \mathrm{Supp}(D)$, which implies that either $a_2 \leqslant 1$ or $a_5 \leqslant 1$, respectively.

Let us show that $L_2+L_2'+L_3\sim -3K_X$. We can easily see that

$$\bar{L}_2\sim_{\mathbb{Q}}\pi^*\big(L_2\big)-\frac{5}{7}E_1-\frac{10}{7}E_2-\frac{8}{7}E_3-\frac{6}{7}E_4-\frac{4}{7}E_5-\frac{2}{7}E_6,$$
$$\bar{L}_2'\sim_{\mathbb{Q}}\pi^*\big(L_2'\big)-\frac{5}{7}E_1-\frac{10}{7}E_2-\frac{8}{7}E_3-\frac{6}{7}E_4-\frac{4}{7}E_5-\frac{2}{7}E_6,$$
$$\bar{L}_3\sim_{\mathbb{Q}}\pi^*\big(L_3\big)-\frac{4}{7}E_1-\frac{8}{7}E_2-\frac{12}{7}E_3-\frac{9}{7}E_4-\frac{6}{7}E_5-\frac{3}{7}E_6,$$

which implies that $L_2+L_2'+L_3\sim_{\mathbb{Q}} -3K_X$, since $\mathrm{Pic}(X)\cong\mathbb{Z}^3$ and

$$L_2\cdot L_2=\frac{3}{7},\ L_2'\cdot L_2'=\frac{3}{7},\ L_3\cdot L_3=\frac{5}{7},\ L_2'\cdot L_3=\frac{8}{7},\ L_2\cdot L_3=\frac{8}{7},\ L_2\cdot L_2'=\frac{10}{7},$$

but $L_2+L_2'+L_3$ is a Cartier divisor, which implies that $L_2+L_2'+L_3\sim -3K_X$.

Since $\mathrm{c}(X, L_2+L_2'+L_3)=1/4$, we may assume that $\mathrm{Supp}(D)$ does not contain at least one curve among L_2, L_2' and L_3 by Remark 2.1, which implies that either $a_2\leqslant 1$ or $a_3\leqslant 1$.

It follows from (4.5) and $a_4\leqslant 2$ that $\mu a_i<1$ for every i. By Lemma 4.6, there exists a point

$$Q\in\Big\{E_2\cap E_3, E_3\cap E_4, E_4\cap E_5\Big\},$$

such that (4.4) is not Kawamata log terminal at the point $Q\in\bar{X}$, but it is Kawamata log terminal elsewhere. Take $k\in\{2,3,4\}$ such that $Q=E_k\cap E_{k+1}$. It follows from Lemma 2.3 that

$$\begin{cases} 2a_k-a_{k-1}-a_{k+1}=\bar{D}\cdot E_k\geqslant \mathrm{mult}_Q\Big(\bar{D}\cdot E_k\Big)>\dfrac{1}{\mu}-a_{k+1}>\dfrac{3}{2}-a_{k+1},\\ 2a_{k+1}-a_k-a_{k+2}=\bar{D}\cdot E_{k+1}\geqslant \mathrm{mult}_Q\Big(\bar{D}\cdot E_{k+1}\Big)>\dfrac{1}{\mu}-a_k\geqslant\dfrac{3}{2}-a_k,\end{cases}$$

which is impossible by (4.5), since $a_4\leqslant 1$, and either $a_2\leqslant 1$ or $a_3\leqslant 1$. □

Lemma 4.12. Suppose that $m=7$. Then the following conditions are equivalent:

- the curve R is irreducible,
- the surface $\bar{X}$ contains an irreducible curve $\bar{L}_4$ such that $\bar{L}_4\cdot\bar{L}_4=-1$ and $\bar{L}_4\cdot E_4=1$.
- the surface $\bar{X}$ contains an irreducible curve $\bar{L}_4$ such that $\bar{L}_4\cdot\bar{L}_4=-1$, $\bar{L}_4\cdot E_4=1$ and
$$\omega\circ\pi\big(\bar{L}_4\big)\subset\mathrm{Supp}\big(R\big).$$

Proof. Suppose that $\bar{X}$ has an irreducible curve $\bar{L}_4$ such that $\bar{L}_4\cdot\bar{L}_4=-1$ and $\bar{L}_4\cdot E_4=1$. Then

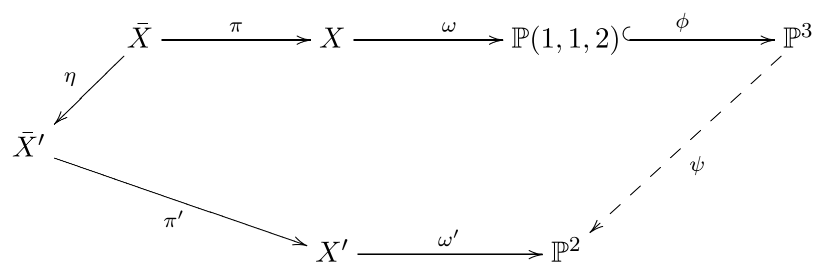
$$\bar{L}_4\sim_{\mathbb{Q}}\pi^*\big(L_4\big)-\frac{1}{2}E_1-E_2-\frac{3}{2}E_3-2E_4-\frac{3}{2}E_5-E_6-\frac{1}{2}E_7,$$

where $L_4=\pi(\bar{L}_4)$. Then $\tau(\bar{L}_4)=\bar{L}_4$ and $\omega(L_4)\subset\mathrm{Supp}(R)$, because

$$-1+\bar{L}_4\cdot\tau\big(\bar{L}_4\big)=\bar{L}_4\cdot\Big(\bar{L}_4+\tau\big(\bar{L}_4\big)\Big)=\bar{L}_4\cdot\Big(\pi^*\big(-2K_X\big)-E_1-2E_2-3E_3-4E_4-3E_5-2E_6-E_7\Big)=-2.$$

Suppose now that the curve R is reducible. Let us show that the surface $\bar{X}$ contains an irreducible curve $\bar{L}_4$ such that $\bar{L}_4\cdot\bar{L}_4=-1$ and $\bar{L}_4\cdot E_4=1$.

Let $\eta\colon\bar{X}\to\bar{X}'$ be a contraction of the curve $\bar{C}$. Then there is a commutative diagram



where π' is a minimal resolution, ϕ is an anticanonical embedding, ψ is a projection from $\phi\circ\omega(P)$, and ω' is a double cover branched at $\psi\circ\phi(R)$. Note that X' is a del Pezzo surface and $K_{X'}^2=2$.

The morphism π' contracts the smooth curves $\eta(E_2)$, $\eta(E_3)$, $\eta(E_4)$, $\eta(E_5)$ and $\eta(E_6)$. But

$$\eta\big(E_2\big)\in\mathrm{Sing}\big(X'\big),$$

and X' has a singularity of type $\mathbb{A}_5$ at the point $\eta(E_2)$. Put $P'=\eta(E_2)$.

Put $R'=\psi\circ\phi(R)$. Then R' is reducible, since R is reducible.

Since $\mathrm{Sing}(\mathbb{P}(1,1,2))\not\in R$, one of the following cases hold:

- either $\phi(R)$ is a union of a smooth conic and an irreducible quartic,
- or the curve $\phi(R)$ is a union of three different smooth conics.

The case when the curve $\phi(R)$ consists of a union of three different smooth conics is impossible, since the surface X' has a singularity of type $\mathbb{A}_5$ at the point $P'=\mathrm{Sing}(X')$.

We see that the curve $\phi(R)$ is a union of a smooth conic and an irreducible quartic curve, which easily implies that R' is a union of a line L and an irreducible cubic curve Z. Then

$$\mathrm{mult}_{\omega'(P')}\Big(L\cdot Z\Big)=3,$$

because X' has a singularity of type $\mathbb{A}_5$ at the point P'. Then $\bar{X}$ contains a curve $\bar{L}_4$ such that

$$\omega'\circ\pi'\circ\eta\big(\bar{L}_4\big)=L,$$

and $\bar{L}_4$ is irreducible. Then $\bar{L}_4\cdot\bar{L}_4=-1$ and $\bar{L}_4\cdot E_4=1$. □

The proof of Lemma 4.12 can be simplified using the results obtained in [31, Section 2].

Lemma 4.13. Suppose that $m=7$ and R is irreducible. Then $\mu\geqslant\mathrm{lct}_3(X)=3/5$.

Proof. Arguing as in the proofs of Lemmas 4.10 and 4.11, we see that there is an irreducible smooth rational curve $\bar{L}_2$ on the surface $\bar{X}$ such that $\bar{L}_2\cdot\bar{L}_2=-1$ and

$$-K_{\bar{X}}\cdot\bar{L}_2=E_2\cdot\bar{L}_2=1,$$

which implies that $E_1\cdot\bar{L}_2=E_3\cdot\bar{L}_2=E_4\cdot\bar{L}_2=E_5\cdot\bar{L}_2=E_6\cdot\bar{L}_2=E_7\cdot\bar{L}_2=\bar{C}\cdot\bar{L}_2=0$.

Put $\bar{L}_5=\tau(\bar{L}_2)$. Then $\bar{L}_5\cdot\bar{L}_5=-1$ and $-K_{\bar{X}}\cdot\bar{L}_5=E_5\cdot\bar{L}_5=1$, which implies that

$$E_1\cdot\bar{L}_5=E_2\cdot\bar{L}_5=E_3\cdot\bar{L}_5=E_4\cdot\bar{L}_5=E_6\cdot\bar{L}_5=E_7\cdot\bar{L}_5=\bar{C}\cdot\bar{L}_5=0.$$

Since the branch curve R is reducible by Lemma 4.12, one can show that there exists an irreducible smooth rational curve $\bar{L}_3$ on the surface $\bar{X}$ such that $\bar{L}_3\cdot\bar{L}_3=-1$ and

$$-K_{\bar{X}}\cdot\bar{L}_3=E_3\cdot\bar{L}_3=1,$$

which implies that $E_1\cdot\bar{L}_3=E_2\cdot\bar{L}_3=E_4\cdot\bar{L}_3=E_5\cdot\bar{L}_3=E_6\cdot\bar{L}_3=E_7\cdot\bar{L}_3=\bar{C}\cdot\bar{L}_3=0$.

Put $\bar{L}_6=\tau(\bar{L}_2)$, $\bar{L}_5=\tau(\bar{L}_3)$, $L_2=\pi(\bar{L}_2)$, $L_3=\pi(\bar{L}_4)$, $L_5=\pi(\bar{L}_5)$ and $L_6=\pi(\bar{L}_6)$. Then

$$\bar{L}_2\sim_{\mathbb{Q}}\pi^*\big(L_2\big)-\frac{3}{4}E_1-\frac{3}{2}E_2-\frac{5}{4}E_3-E_4-\frac{3}{4}E_5-\frac{1}{2}E_6-\frac{1}{4}E_7,$$
$$\bar{L}_3\sim_{\mathbb{Q}}\pi^*\big(L_3\big)-\frac{5}{8}E_1-\frac{5}{4}E_2-\frac{15}{8}E_3-\frac{3}{2}E_4-\frac{9}{8}E_5-\frac{3}{4}E_6-\frac{3}{8}E_7,$$
$$\bar{L}_5\sim_{\mathbb{Q}}\pi^*\big(L_5\big)-\frac{3}{8}E_1-\frac{3}{4}E_2-\frac{9}{8}E_3-\frac{3}{2}E_4-\frac{15}{8}E_5-\frac{5}{4}E_6-\frac{5}{8}E_7,$$
$$\bar{L}_6\sim_{\mathbb{Q}}\pi^*\big(L_6\big)-\frac{1}{4}E_1-\frac{1}{2}E_2-\frac{3}{4}E_3-E_4-\frac{5}{4}E_5-\frac{3}{2}E_6-\frac{3}{4}E_7,$$

which implies that $L_2+2L_3\sim-3K_X$. Indeed, we have $L_2+2L_3\sim_{\mathbb{Q}}-3K_X$, since

$$L_2\cdot L_2=\frac{1}{2},\ L_3\cdot L_3=\frac{7}{8},\ L_2\cdot L_3=\frac{5}{4},$$

and $\mathrm{Pic}(X)\cong\mathbb{Z}^3$. But L_2+2L_3 is a Cartier divisor, which implies that $L_2+2L_3\sim-3K_X$.

We have $\mathrm{c}(X,L_2+2L_3)=3/15$ and $L_2+2L_3\sim-3K_X$, which implies that $\mathrm{lct}_3(X)\leqslant3/5$.

To complete the proof, it is enough to show that $\mu \geqslant 3/5$.

Suppose that $\mu < 3/5$. Let us derive a contradiction.

By Remark 2.1, we may assume that the support of the divisor $\bar{D}$ does not contain at least one components of every curve $\bar{L}_2 + \bar{L}_6$, $\bar{L}_2 + 2\bar{L}_3$, $\bar{L}_3 + \bar{L}_5$. But

$$\bar{D} \cdot \bar{L}_i = 1 - a_i,$$

which implies that $a_i \leqslant 1$ if $\bar{L}_i \not\subset \mathrm{Supp}(\bar{D})$. Therefore, either $a_3 \leqslant 1$ or $a_2 \leqslant 1$ and $a_5 \leqslant 1$.

If $a_3 \leqslant 1$, then it follows from (4.5) that

$$a_1 \leqslant \frac{7}{8},\ a_2 \leqslant \frac{6}{5},\ a_3 \leqslant 1,\ a_4 \leqslant \frac{4}{3},\ a_5 \leqslant \frac{5}{3},\ a_6 \leqslant \frac{3}{2},\ a_7 \leqslant \frac{7}{8}.$$

If $a_2 \leqslant 1$ and $a_5 \leqslant 1$, then it follows from (4.5) that

$$a_1 \leqslant \frac{7}{8},\ a_2 \leqslant 1,\ a_3 \leqslant \frac{3}{2},\ a_4 \leqslant \frac{4}{3},\ a_5 \leqslant 1,\ a_6 \leqslant \frac{6}{5},\ a_7 \leqslant \frac{7}{8}.$$

By Lemma 4.6, there exists $k \in \{2,3,4,5\}$ such that (4.4) is not Kawamata log terminal at the point $E_k \cap E_{k+1}$ and is Kawamata log terminal outside of $E_k \cap E_{k+1}$.

Put $Q = E_k \cap E_{k+1}$. Then it follows from Lemma 2.3 that

$$\begin{cases} 2a_k - a_{k-1} - a_{k+1} = \bar{D} \cdot E_k \geqslant \mathrm{mult}_Q\Big(\bar{D} \cdot E_k\Big) > \dfrac{1}{\mu} - a_{k+1} > \dfrac{5}{3} - a_{k+1}, \\ 2a_{k+1} - a_k - a_{k+2} = \bar{D} \cdot E_{k+1} \geqslant \mathrm{mult}_Q\Big(\bar{D} \cdot E_{k+1}\Big) > \dfrac{1}{\mu} - a_k \geqslant \dfrac{5}{3} - a_k, \end{cases}$$

which is impossible by (4.5), since we assume that either $a_3 \leqslant 1$ or $a_2 \leqslant 1$ and $a_5 \leqslant 1$. □

Lemma 4.14. Suppose that $m = 7$ and R is reducible. Then $\mu \geqslant \mathrm{lct}_2(X) = 1/2$.

Proof. By Lemma 4.12, the surface X contains an irreducible curve $\bar{L}_4$ such that

$$\omega \circ \pi\big(\bar{L}_4\big) \subset \mathrm{Supp}\big(R\big)$$

and $-\bar{L}_4 \cdot \bar{L}_4 = \bar{L}_4 \cdot E_4 = 1$. Then $-K_{\bar{X}} \cdot \bar{L}_4 = 1$, which implies that

$$E_1 \cdot \bar{L}_4 = E_2 \cdot \bar{L}_4 = E_3 \cdot \bar{L}_4 = E_5 \cdot \bar{L}_4 = E_6 \cdot \bar{L}_4 = E_7 \cdot \bar{L}_4 = \bar{C} \cdot \bar{L}_4 = 0.$$

Put $L_4 = \pi(\bar{L}_4)$. Then $2L_4 \sim -2K_X$ and

$$\bar{L}_4 \sim_{\mathbb{Q}} \pi^*\big(L_4\big) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - 2E_4 - \frac{3}{2}E_5 - E_6 - \frac{1}{2}E_7,$$

which implies that $\mathrm{lct}_2(X) \leqslant \mathrm{c}(X, L_4) = 1/2$.

To complete the proof, it is enough to show that $\mu \geqslant 1/2$.

Suppose that $\mu < 1/2$. Let us derive a contradiction.

By Remark 2.1, we may assume that $L_4 \not\subset \mathrm{Supp}(D)$. Then

$$0 \leqslant \bar{L}_4 \cdot \bar{D} = 1 - a_4,$$

which implies that $a_4 \leqslant 1$. Thus, it follows from (4.5) that

$$a_1 \leqslant \frac{7}{8},\ a_2 \leqslant \frac{3}{2},\ a_3 \leqslant \frac{5}{4},\ a_4 \leqslant 1,\ a_5 \leqslant \frac{5}{4},\ a_6 \leqslant \frac{3}{2},\ a_7 \leqslant \frac{7}{8}.$$

It follows from Lemma 4.6 that there exists a point

$$Q \in \Big\{E_2 \cap E_3, E_3 \cap E_4, E_4 \cap E_5, E_5 \cap E_6\Big\}$$

such that $\mathrm{LCS}(\bar{X}, \mu\bar{D} + \sum_{i=1}^{7} \mu a_i E_i) = Q$.

Without loss of generality, we may assume that either $Q = E_2 \cap E_3$ or $Q = E_3 \cap E_4$.

If $Q = E_3 \cap E_4$, then it follows from Lemma 2.3 that

$$2a_4 - a_3 - a_5 = \bar{D} \cdot E_4 \geqslant \mathrm{mult}_Q\Big(\bar{D} \cdot E_4\Big) > \frac{1}{\mu} - a_3 > 2 - a_3,$$

which together with (4.5) imply that $a_4 > 1$, which is a contradiction.

If $Q = E_2 \cap E_3$, then it follows from Lemma 2.3 that

$$2a_3 - a_2 - a_4 = \bar{D} \cdot E_3 \geqslant \mathrm{mult}_Q\Big(\bar{D} \cdot E_3\Big) > \frac{1}{\mu} - a_2 > 2 - a_2,$$

which together with (4.5) immediately leads to a contradiction. □

Lemma 4.15. Suppose that $m = 8$. Then $\mu \geqslant \mathrm{lct}_3(X) = 1/2$.

Proof. Arguing as in the proofs of Lemmas 4.10 and 4.11, we see that there is an irreducible smooth rational curve $\bar{L}_3$ on the surface $\bar{X}$ such that $\bar{L}_3 \cdot \bar{L}_3 = -1$ and

$$-K_{\bar{X}} \cdot \bar{L}_3 = E_3 \cdot \bar{L}_3 = 1,$$

which implies that $E_1 \cdot \bar{L}_3 = E_2 \cdot \bar{L}_3 = E_4 \cdot \bar{L}_3 = E_5 \cdot \bar{L}_3 = E_6 \cdot \bar{L}_3 = E_7 \cdot \bar{L}_3 = \bar{C} \cdot \bar{L}_3 = 0$.

Put $\bar{L}_6 = \tau(\bar{L}_3)$. Then $\bar{L}_6 \cdot \bar{L}_6 = -1$ and $-K_{\bar{X}} \cdot \bar{L}_6 = E_6 \cdot \bar{L}_6 = 1$, which implies that

$$E_1 \cdot \bar{L}_6 = E_2 \cdot \bar{L}_6 = E_3 \cdot \bar{L}_6 = E_4 \cdot \bar{L}_6 = E_5 \cdot \bar{L}_6 = E_7 \cdot \bar{L}_6 = \bar{C} \cdot \bar{L}_6 = 0.$$

Put $L_3 = \pi(\bar{L}_3)$ and $L_6 = \pi(\bar{L}_6)$. Then $3L_3 \sim 3L_6 \sim -3K_X$. On the other hand, we have

$$\bar{L}_3 \sim_{\mathbb{Q}} \pi^*\big(L_3\big) - \frac{2}{3}E_1 - \frac{4}{3}E_2 - 2E_3 - \frac{5}{3}E_4 - \frac{4}{3}E_5 - E_6 - \frac{2}{3}E_7 - \frac{1}{3}E_8,$$

$$\bar{L}_6 \sim_{\mathbb{Q}} \pi^*\big(L_6\big) - \frac{1}{3}E_1 - \frac{2}{3}E_2 - E_3 - \frac{4}{3}E_4 - \frac{5}{3}E_5 - 2E_6 - \frac{4}{3}E_7 - \frac{2}{3}E_8,$$

which implies $\mathrm{c}(X, L_3) = \mathrm{c}(X, L_6) = 1/2$. Then $\mathrm{lct}_3(X) \leqslant 1/2$.

To complete the proof, it is enough to show that $\mu \geqslant 1/2$.

Suppose that $\mu < 1/2$. Let us derive a contradiction.

By Remark 2.1, we may assume that $\mathrm{Supp}(\bar{D})$ does not contain $\bar{L}_3$ and $\bar{L}_6$. Then

$$1 - a_3 = \bar{D} \cdot \bar{L}_3 \geqslant 0,$$

which implies that $a_3 \leqslant 1$. Similarly, we have $a_6 \leqslant 1$. Then it follows from (4.5) that

$$a_1 \leqslant \frac{8}{9},\ a_2 \leqslant \frac{7}{6},\ a_3 \leqslant 1,\ a_4 \leqslant \frac{4}{3}, a_5 \leqslant \frac{4}{3},\ a_6 \leqslant 1,\ a_7 \leqslant \frac{7}{6},\ a_8 \leqslant \frac{8}{9}.$$

By Lemma 4.6, there exists $k \in \{2, 3, 4, 5, 6\}$ such that (4.4) is not Kawamata log terminal at the point $E_k \cap E_{k+1}$ and is Kawamata log terminal outside of the point $E_k \cap E_{k+1}$.

Put $Q = E_k \cap E_{k+1}$. Then it follows from Lemma 2.3 that

$$\begin{cases} 2a_k - a_{k-1} - a_{k+1} = \bar{D} \cdot E_k \geqslant \mathrm{mult}_Q\Big(\bar{D} \cdot E_k\Big) > \dfrac{1}{\mu} - a_{k+1} > \dfrac{1}{2} - a_{k+1}, \\ 2a_{k+1} - a_k - a_{k+2} = \bar{D} \cdot E_{k+1} \geqslant \mathrm{mult}_Q\Big(\bar{D} \cdot E_{k+1}\Big) > \dfrac{1}{\mu} - a_k \geqslant \dfrac{1}{2} - a_k, \end{cases}$$

which is impossible by (4.5), since $a_3 \leqslant 1$ and $a_6 \leqslant 1$. □

The assertion of Theorem 4.1 is proved.

5. One non-cyclic singular point

Let X be a sextic surface in $\mathbb{P}(1,1,2,3)$ with canonical singularities such that $|\mathrm{Sing}(X)| = 1$, and $\mathrm{Sing}(X)$ consists of a singular point of type $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{D}_6$, $\mathbb{D}_7$, $\mathbb{D}_8$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$.

Theorem 5.1. The following equality holds:

$$\mathrm{lct}\big(X\big) = \begin{cases} \mathrm{lct}_2\big(X\big) = 1/3 \text{ if } P \text{ is a point of type } \mathbb{D}_8, \\ \mathrm{lct}_2\big(X\big) = 2/5 \text{ if } P \text{ is a point of type } \mathbb{D}_7, \\ \mathrm{lct}_1\big(X\big) \text{ in the remaining cases.} \end{cases}$$

Corollary 5.2. The inequality $\mathrm{lct}(X) \leqslant 1/2$ holds.

In the rest of this section we will prove Theorem 5.1.

Let D be an effective $\mathbb{Q}$-divisor on X such that $D \sim_{\mathbb{Q}} -K_X$. We must show that

$$\mathrm{c}\big(X, D\big) \geqslant \begin{cases} \mathrm{lct}_2\big(X\big) = 1/3 \text{ if } P \text{ is a point of type } \mathbb{D}_8, \\ \mathrm{lct}_2\big(X\big) = 2/5 \text{ if } P \text{ is a point of type } \mathbb{D}_7, \\ \mathrm{lct}_1\big(X\big) \text{ in the remaining cases.} \end{cases}$$

to prove Theorem 5.1. Put $\mu = \mathrm{c}(X, D)$.

Suppose that $\mu < \mathrm{lct}_1(X)$. Then $\mathrm{LCS}(X, \mu D) = \mathrm{Sing}(X)$ by Lemma 2.6. Put $P = \mathrm{Sing}(X)$.

Let $\pi\colon \bar{X} \to X$ be a minimal resolution, let $E_1, E_2 \ldots, E_m$ be irreducible π-exceptional curves, let C be the curve in $|-K_X|$ such that $P \in C$, and let $\bar{C}$ be its proper transform on $\bar{X}$. Then

$$\bar{C} \sim_{\mathbb{Q}} \pi^*\big(C\big) - \sum_{i=1}^{m} n_i E_i,$$

where $n_i \in \mathbb{N}$. Without loss of generality, we may assume that $E_3 \cdot \sum_{i\neq 3} E_i = 3$. Then

$$\mathrm{lct}_1\big(X\big) = \mathrm{c}\big(X, C\big) = \frac{1}{n_3} = \begin{cases} 1/2 \text{ if } P \text{ is of type } \mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{D}_7 \text{ or } \mathbb{D}_8, \\ 1/3 \text{ if } P \text{ is of type } \mathbb{E}_6, \\ 1/4 \text{ if } P \text{ is of type } \mathbb{E}_7, \\ 1/6 \text{ if } P \text{ is of type } \mathbb{E}_8. \end{cases}$$

By Remark 2.1, we may assume that $C \not\subset \mathrm{Supp}(D)$, since the curve C is irreducible.

Let $\bar{D}$ be the proper transform of the divisor D on the surface $\bar{X}$. Then

$$\bar{D} \sim_{\mathbb{Q}} \pi^*\big(D\big) - \sum_{i=1}^{m} a_i E_i,$$

where a_i is a non-negative rational number. Then

$$K_{\bar{X}} + \mu\Big(\bar{D} + \sum_{i=1}^{m} a_i E_i\Big) \sim_{\mathbb{Q}} \pi^*\Big(K_X + \mu D\Big),$$

which implies that $(\bar{X}, \mu\bar{D} + \sum_{i=1}^{m} \mu a_i E_i)$ is not Kawamata log terminal (see Remark 2.4).

Lemma 5.3. The equality $\mu a_3 = 1$ holds.

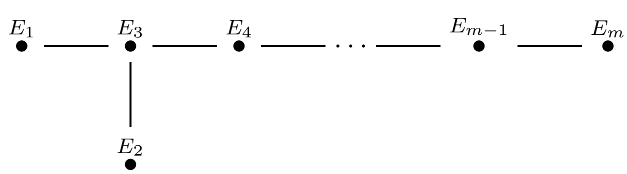
Proof. The equality $\mu a_3 = 1$ follows from Lemma 2.5. □

Lemma 5.4. Suppose that P is not a point of type $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$. Then

$$\mu \geqslant \begin{cases} \mathrm{lct}_2\big(X\big) = 1/3 \text{ if } P \text{ is a point of type } \mathbb{D}_8, \\ \mathrm{lct}_2\big(X\big) = 2/5 \text{ if } P \text{ is a point of type } \mathbb{D}_7, \end{cases}$$

and P is either a point of type $\mathbb{D}_7$ or is a point of type $\mathbb{D}_8$.

Proof. Without loss of generality, we may assume that the diagram



shows how the π-exceptional curves intersect each other. Then

$$\bar{C}\sim_{\mathbb{Q}}\pi^{*}\big(C\big)-E_{1}-E_{2}-E_{m}-\sum_{i=3}^{m-1}2E_{i},$$

which implies that $\bar{C}\cdot E_{m-1}=1$ and $\bar{C}\cdot E_{i}=0\iff i\neq m-1$. Then

$$\tag{5.5}\begin{cases}1-a_{m-1}=\bar{D}\cdot\bar{C}\geqslant 0,\\ 2a_{1}-a_{3}=\bar{D}\cdot E_{1}\geqslant 0,\\ 2a_{2}-a_{3}=\bar{D}\cdot E_{2}\geqslant 0,\\ 2a_{3}-a_{1}-a_{2}-a_{3}=\bar{D}\cdot E_{3}\geqslant 0,\\ \cdots\\ 2a_{m-1}-a_{m-2}-a_{m}=\bar{D}\cdot E_{m-1}\geqslant 0,\\ 2a_{m}-a_{m-1}=\bar{D}\cdot E_{m}\geqslant 0,\end{cases}$$

which easily implies that $a_3\leqslant 2$ if $m\leqslant 6$. But $\mu a_3=1$ and $\mu<\mathrm{lct}_1(X)=1/2$ by Lemma 5.3, which implies that either $m=7$ or $m=8$.

Arguing as in the proofs of Lemmas 4.10 and 4.11, we may assume that there is an irreducible smooth rational curve $\bar{L}_1$ on the surface $\bar{X}$ such that $\bar{L}_1\cdot\bar{L}_1=-1$ and

$$-K_{\bar{X}}\cdot\bar{L}_{1}=E_{1}\cdot\bar{L}_{1}=1,$$

which implies that $\bar{C}\cdot\bar{L}_1=0$ and $E_i\cdot\bar{L}_1=0\iff i\neq 1$.

Let $\omega\colon X\to\mathbb{P}(1,1,2)$ be the natural double cover given by $|-2K_X|$, and let τ be a biregular involution of the surface $\bar{X}$ that is induced by ω. Put $\bar{L}_2=\tau(\bar{L}_1)$. If $m=7$, then

$$-K_{\bar{X}}\cdot\bar{L}_{2}=E_{2}\cdot\bar{L}_{2}=1$$

and $\bar{L}_2\cdot\bar{L}_2=-1$, which implies that $\bar{C}\cdot\bar{L}_2=0$ and $E_i\cdot\bar{L}_2=0\iff i\neq 2$.

Put $L_1=\pi(\bar{L}_1)$ and $L_2=\pi(\bar{L}_2)$. Then $L_1+L_2\sim -2K_X$. If $m=7$, then

$$\bar{L}_{1}\sim_{\mathbb{Q}}\pi^{*}\big(L_{1}\big)-\frac{7}{4}E_{1}-\frac{5}{4}E_{2}-\frac{5}{2}E_{3}-2E_{4}-\frac{3}{2}E_{5}-E_{6}-\frac{1}{2}E_{7},$$
$$\bar{L}_{2}\sim_{\mathbb{Q}}\pi^{*}\big(L_{2}\big)-\frac{5}{4}E_{1}-\frac{7}{4}E_{2}-\frac{5}{2}E_{3}-2E_{4}-\frac{3}{2}E_{5}-E_{6}-\frac{1}{2}E_{7},$$

which implies that $\mathrm{c}(X,L_1+L_2)=1/5$ and $\mathrm{lct}_2(X)\leqslant 2/5$. If $m=7$, then

$$a_{3}\leqslant\frac{5}{2}$$

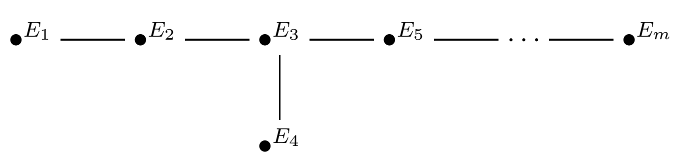
by (5.5). But $\mu a_3=1$ by Lemma 5.3. Then $\mu\geqslant 2/5$ if $m=7$, which is exactly what we need.

We may assume that $m=8$. Then $\bar{L}_2=\bar{L}_1$ and

$$\bar{L}_{1}\sim_{\mathbb{Q}}\pi^{*}\big(L_{1}\big)-2E_{1}-\frac{3}{2}E_{2}-3E_{3}-\frac{5}{2}E_{4}-2E_{5}-\frac{3}{2}E_{6}-E_{7}-\frac{1}{2}E_{8},$$

which implies that $\mathrm{lct}_2(X)\leqslant\mathrm{c}(X,L_1)=1/3$. But $a_3\leqslant 1/3$ by (5.5) and $\mu a_3=1$ by Lemma 5.3, which implies that $\mu\geqslant 1/3$, which completes the proof since $\mathrm{lct}_2(X)\geqslant\mathrm{lct}(X)$. □

To complete the proof of Theorem 5.1, we may assume that P is a point of type $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$. Without loss of generality, we may assume that the diagram



shows how the π-exceptional curves intersect each other. It is well-known (cf. [29][30]) that

- if $m=6$, then $\bar{C}\cdot E_4=1$, which implies that and $\bar{C}\cdot E_i=0 \iff i\neq 4$,
- if $m=7$, then $\bar{C}\cdot E_1=1$, which implies that and $\bar{C}\cdot E_i=0 \iff i\neq 1$,
- if $m=8$, then $\bar{C}\cdot E_8=1$, which implies that and $\bar{C}\cdot E_i=0 \iff i\neq 8$.

Put $k=4$ if $m=6$, put $k=1$ if $m=7$, put $k=8$ if $m=8$. Then

$$\begin{cases} 1-a_k=\bar{D}\cdot\bar{C}\geqslant 0,\\ 2a_1-a_3=\bar{D}\cdot E_1\geqslant 0,\\ 2a_2-a_3-a_1=\bar{D}\cdot E_2\geqslant 0,\\ 2a_3-a_2-a_4-a_5=\bar{D}\cdot E_3\geqslant 0,\\ 2a_4-a_3=\bar{D}\cdot E_4\geqslant 0,\\ 2a_5-a_3-a_6=\bar{D}\cdot E_5\geqslant 0,\\ \cdots\\ 2a_{m-1}-a_{m-2}-a_m=\bar{D}\cdot E_{m-1}\geqslant 0,\\ 2a_m-a_{m-1}=\bar{D}\cdot E_m\geqslant 0,\end{cases}\tag{5.6}$$

which implies that $a_3<n_3$. But $n_3=1/\mathrm{lct}_1(X)$ and $\mu a_3=1$ by Lemma 5.3. Then $\mu\geqslant\mathrm{lct}_1(X)$.

The assertion of Theorem 5.1 is proved.

6. Many singular points

Let X be a sextic surface in $\mathbb{P}(1,1,2,3)$ with canonical singularities such that $|\mathrm{Sing}(X)|\geqslant 2$.

Theorem 6.1. The following equality holds:

$$\mathrm{lct}\big(X\big)=\begin{cases}\mathrm{lct}_2\big(X\big)=1/2 \text{ if } \mathrm{Sing}(X) \text{ consists of a point of type } \mathbb{A}_7 \text{ and a point of type } \mathbb{A}_1,\\ \mathrm{lct}_2\big(X\big)=2/3 \text{ if } X \text{ has a singular point of type } \mathbb{A}_6,\\ \mathrm{lct}_2\big(X\big)=2/3 \text{ if } X \text{ has a singular point of type } \mathbb{A}_5,\\ \mathrm{lct}_2\big(X\big)=\min\big(\mathrm{lct}_1\big(X\big),4/5\big) \text{ if } X \text{ has a singular point of type } \mathbb{A}_4,\\ \mathrm{lct}_1\big(X\big) \text{ in the remaining cases,}\end{cases}$$

and if there exists an effective $\mathbb{Q}$-divisor D on the surface X such that $D\sim_{\mathbb{Q}}-K_X$ and

$$\mathrm{c}\big(X,D\big)=\mathrm{lct}\big(X\big)=\frac{2}{3},$$

then either D is an irreducible curve in $|-K_X|$ with a cusp at a point in $\mathrm{Sing}(X)$ of type $\mathbb{A}_2$, or the divisor D is uniquely defined and it can be explicitly described.

Let D be an arbitrary effective $\mathbb{Q}$-divisor on the surface X such that

$$D\sim_{\mathbb{Q}}-K_X,$$

and put $\mu=\mathrm{c}(X,D)$. To prove Theorem 6.1, it is enough to show that

$$\mu\geqslant\begin{cases}\mathrm{lct}_2\big(X\big)=1/2 \text{ if } \mathrm{Sing}(X) \text{ consists of a point of type } \mathbb{A}_7 \text{ and a point of type } \mathbb{A}_1,\\ \mathrm{lct}_2\big(X\big)=2/3 \text{ if } X \text{ has a singular point of type } \mathbb{A}_6,\\ \mathrm{lct}_2\big(X\big)=2/3 \text{ if } X \text{ has a singular point of type } \mathbb{A}_5,\\ \mathrm{lct}_2\big(X\big)=\min\big(\mathrm{lct}_1\big(X\big),4/5\big) \text{ if } X \text{ has a singular point of type } \mathbb{A}_4,\\ \mathrm{lct}_1\big(X\big) \text{ in the remaining cases,}\end{cases}$$

and if $\mu=\mathrm{lct}(X)=2/3$, then we have the following two possibilities:

- either D is a curve in $|-K_X|$ with a cusp at a point in $\mathrm{Sing}(X)$ of type $\mathbb{A}_2$,
- or the divisor D is uniquely defined and it can be explicitly described.

Lemma 6.2. If $\mathrm{Sing}(X)$ has a point of type $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{D}_6$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$, then $\mu \geqslant \mathrm{lct}_1(X)$.

Proof. Suppose that $\mathrm{Sing}(X)$ has a point of type $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{D}_6$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$, but $\mu < \mathrm{lct}_1(X)$. Then

$$\mathrm{LCS}\big(X, \mu D\big) \subsetneq \mathrm{Sing}\big(X\big)$$

and $\mathrm{LCS}(X, \mu D)$ consists of a point in $\mathrm{Sing}(X)$ that is not of type $\mathbb{A}_1$ or $\mathbb{A}_2$ by Lemma 2.6.

If the locus $\mathrm{LCS}(X, \mu D)$ is a singular point of the surface X of type $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{D}_6$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$, then arguing as in the proof of Theorem 5.1, we immediately obtain a contradiction.

By Remark 1.22, the locus $\mathrm{LCS}(X, \mu D)$ must be a singular point of the surface X of type $\mathbb{A}_3$, and we can easily obtain a contradiction arguing as in the proof of Corollary 4.7. □

Lemma 6.3. Suppose that $\mathrm{Sing}(X)$ consists of points of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$. Then $\mu \geqslant \mathrm{lct}_1(X)$. If

$$\mu = \mathrm{lct}_1\big(X\big) = \frac{2}{3},$$

then D is a curve in $|-K_X|$ with a cusp at a point in $\mathrm{Sing}(X)$ of type $\mathbb{A}_2$.

Proof. This follows from Lemma 2.6 and the proof of Corollary 4.7. □

By Remark 1.22 and Lemmas 6.2 and 6.2, we may assume that

$$\mathrm{Sing}\big(X\big) \in \left\{\begin{array}{l} \mathbb{A}_7+\mathbb{A}_1, \mathbb{A}_6+\mathbb{A}_1, \mathbb{A}_5+\mathbb{A}_1, \mathbb{A}_5+\mathbb{A}_1+\mathbb{A}_1, \mathbb{A}_5+\mathbb{A}_2, \mathbb{A}_5+\mathbb{A}_2+\mathbb{A}_1,\\ \mathbb{A}_4+\mathbb{A}_4, \mathbb{A}_4+\mathbb{A}_3, \mathbb{A}_4+\mathbb{A}_2+\mathbb{A}_1, \mathbb{A}_4+\mathbb{A}_2, \mathbb{A}_4+\mathbb{A}_1+\mathbb{A}_1, \mathbb{A}_4+\mathbb{A}_1, \end{array}\right\},$$

which implies that there is a point $P \in \mathrm{Sing}(X)$ that is a point of type $\mathbb{A}_m$ for $m \in \{4,5,6,7\}$.

Let $\pi\colon \bar{X} \to X$ be a minimal resolution, let $E_1, E_2, \ldots, E_m$ be π-exceptional curves such that

$$E_i \cdot E_j \neq 0 \iff \big|i-j\big| \leqslant 1$$

and $\pi(E_i) = P$ for every $i \in \{1, \ldots, m\}$, let C be the unique curve in $|-K_X|$ such that $P \in C$, and let $\bar{C}$ be the proper transform of the curve C on the surface $\bar{X}$. Then

$$\bar{C} \cdot E_1 = \bar{C} \cdot E_m = 1,$$

and $\bar{C} \cdot E_2 = \bar{C} \cdot E_3 = \cdots = \bar{C} \cdot E_{m-1} = 0$. Note that $\bar{C} \cong \mathbb{P}^1$ and $\bar{C} \cdot \bar{C} = -1$.

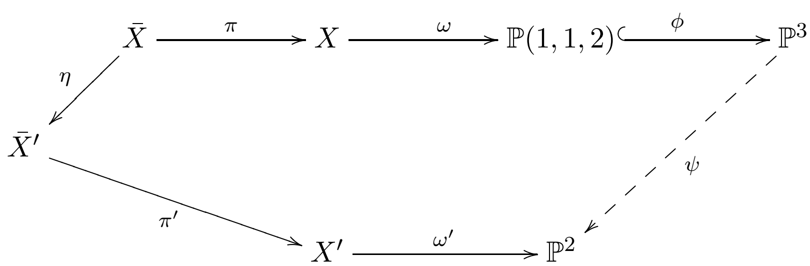
Let $\bar{D}$ be the proper transform of D on the surface $\bar{X}$. Then

$$\bar{D} \sim_{\mathbb{Q}} \pi^*\big(D\big) - \sum_{i=1}^{m} a_i E_i,$$

where a_i is a non-negative rational number. Then

$$\left\{\begin{array}{l} 1 - a_1 - a_m = \bar{D} \cdot \bar{C} \geqslant 0,\\ 2a_1 - a_2 = \bar{D} \cdot E_1 \geqslant 0,\\ \cdots\\ 2a_{m-1} - a_{m-2} - a_m = \bar{D} \cdot E_{m-1} \geqslant 0,\\ 2a_m - a_{m-1} = \bar{D} \cdot E_m \geqslant 0, \end{array}\right. \tag{6.4}$$

Let $\eta\colon \bar{X} \to \bar{X}'$ be a contraction of the curve $\bar{C}$. Then there is a commutative diagram



where ω and ω' are natural double covers π' is a minimal resolution, ϕ is an anticanonical embedding, and ψ is a projection from $\phi\circ\omega(P)$. Put $P'=\eta(E_2)$. Then $P'\in\mathrm{Sing}(X')$.

Remark 6.5. The birational morphism π' contracts the smooth curves $\eta(E_2),\eta(E_3),\ldots,\eta(E_{m-1})$, and $\pi'\circ\eta$ contracts all π-exceptional curves that are different from the curves $E_1,E_2,\ldots,E_m$.

Let R be the branch curve in $\mathbb{P}(1,1,2)$ of the double cover ω. Put $R'=\psi\circ\phi(R)$.

Lemma 6.6. Suppose that $m=7$. Then $\mu\geqslant\mathrm{lct}_2(X)=1/2$.

Proof. Let $\alpha\colon\bar{X}\to\breve{X}$ be a contraction of the irreducible curves $\bar{C}$, E_7, E_6, E_5, E_4, E_3 and E_2, and let F be the π-exceptional curve such that $\pi(F)$ is a point of type $\mathbb{A}_1$. Then

$$\breve{X}\cong\mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(2)\Big).$$

Let $\breve{L}_2$ be the fiber of the projection $\breve{X}\to\mathbb{P}^1$ such that $\alpha(\bar{C})\in\breve{L}_2$, and let $\bar{L}_2$ be the proper transform of the curve $\breve{L}_2$ on the surface $\bar{X}$ via α. Then $\bar{L}_2\cdot\bar{L}_2=-1$ and

$$-K_{\bar{X}}\cdot\bar{L}_2=E_2\cdot\bar{L}_2=F\cdot\bar{L}_2=1,$$

which implies that $E_1\cdot\bar{L}_2=E_3\cdot\bar{L}_2=E_4\cdot\bar{L}_2=E_5\cdot\bar{L}_2=E_6\cdot\bar{L}_2=E_7\cdot\bar{L}_2=\bar{C}\cdot\bar{L}_2=0$.

Let $\beta\colon\bar{X}\to\check{X}$ be a contraction of the curves $\bar{L}_2$, E_2, $\bar{C}$, E_7, E_6, E_5, E_4. Then

$$\beta\big(E_3\big)\cdot\beta\big(E_3\big)=\beta\big(F\big)\cdot\beta\big(F\big)=0,$$

and $\check{X}$ is a smooth del Pezzo surface such that $K_{\check{X}}^2=8$. Then $\check{X}\cong\mathbb{P}^1\times\mathbb{P}^1$.

Let $\check{L}_4$ be the curve in $|\beta(F)|$ such that $\beta(E_4)\in\check{L}_4$, and let $\bar{L}_3$ be its proper transform on the surface $\bar{X}$ via β. Then one can easily check that $\bar{L}_4\cdot\bar{L}_4=-1$ and

$$-K_{\bar{X}}\cdot\bar{L}_4=E_4\cdot\bar{L}_4=1,$$

which implies that $E_1\cdot\bar{L}_4=E_2\cdot\bar{L}_4=E_3\cdot\bar{L}_4=E_5\cdot\bar{L}_4=E_6\cdot\bar{L}_4=E_7\cdot\bar{L}_4=\bar{C}\cdot\bar{L}_4=F\cdot\bar{L}_4=0$.

Put $L_4=\pi(\bar{L}_4)$. Then one can easily check that

$$\bar{L}_4\sim_{\mathbb{Q}}\pi^*\big(L_4\big)-\frac{1}{2}E_1-E_2-\frac{3}{2}E_3-2E_4-\frac{3}{2}E_5-E_6-\frac{1}{2}E_7,$$

which implies that $\mathrm{c}(X,L_4)=1/2$. But $2L_4\sim-2K_X$, which implies that $\mathrm{lct}_2(X)\leqslant1/2$.

Arguing as in the proof of Lemma 4.12, we see that $\omega(L_4)\subset\mathrm{Supp}(R)$.

Arguing as in the proof of Lemma 4.14 and using (6.4), we see that $\mu\geqslant\mathrm{lct}_2(X)=1/2$. □

Lemma 6.7. Suppose that $m=6$. Then $\mu\geqslant\mathrm{lct}_2(X)=2/3$, and if $\mu=2/3$, then

- either D is a curve in $|-K_X|$ with a cusp at a point in $\mathrm{Sing}(X)$ of type $\mathbb{A}_2$,
- or the divisor D is uniquely defined and can be explicitly described.

Proof. Let $\alpha\colon\bar{X}\to\breve{X}$ be a contraction of the curves $\bar{C}$, E_6, E_5, E_4, E_3, E_2. Then $\breve{X}$ is a smooth surface such that $K_{\breve{X}}^2=7$, and $-K_X$ is nef. There is a birational morphism $\gamma\colon\breve{X}\to\hat{X}$ such that

$$\hat{X}\cong\mathbb{P}\Big(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(2)\Big),$$

and γ is a blow down of a smooth irreducible rational curve that does not contain the point $\alpha(\bar{C})$.

Let $\hat{L}_2$ be the fiber of the projection $\hat{X}\to\mathbb{P}^1$ such that $\gamma\circ\alpha(\bar{C})\in\hat{L}_2$, and let $\bar{L}_2$ be the proper transform of the curve $\hat{L}_2$ on the surface $\bar{X}$ via $\gamma\circ\alpha$. Then $\bar{L}_2\cdot\bar{L}_2=-1$ and

$$-K_{\bar{X}}\cdot\bar{L}_2=E_2\cdot\bar{L}_2=1,$$

which implies that $E_1\cdot\bar{L}_2=E_3\cdot\bar{L}_2=E_4\cdot\bar{L}_2=E_5\cdot\bar{L}_2=E_6\cdot\bar{L}_2=\bar{C}\cdot\bar{L}_2=0$.

Let $\beta\colon\bar{X}\to\check{X}$ be a contraction of the curves $\bar{L}_2$, $\bar{C}$, E_6, E_5, E_4, and let F be the π-exceptional curve such that $\pi(F)$ is a point of type $\mathbb{A}_1$. Then

$$\beta\big(E_2\big)\cdot\beta\big(E_2\big)=\beta\big(E_3\big)\cdot\beta\big(E_3\big)=\beta\big(F\big)\cdot\beta\big(F\big)=-1,$$

and $\check{X}$ is a smooth del Pezzo surface such that $K^2_{\check{X}}=6$. Thus, there exists an irreducible smooth rational curve $\check{L}_3$ on the surface $\check{X}$ such that $\check{L}_3\cdot\check{L}_3=-1$, $\check{L}_3\cdot\beta(E_3)=1$ and $\check{L}_3\cdot\beta(F)=0$.

Let $\bar{L}_3$ be the proper transform of the curve $\check{L}_3$ on the surface $\bar{X}$. Then $\bar{L}_3\cdot\bar{L}_3=-1$ and

$$-K_{\bar{X}}\cdot\bar{L}_3=E_3\cdot\bar{L}_3=1,$$

which implies that $E_1\cdot\bar{L}_3=E_2\cdot\bar{L}_3=E_4\cdot\bar{L}_3=E_5\cdot\bar{L}_3=E_6\cdot\bar{L}_3=\bar{C}\cdot\bar{L}_3=F\cdot\bar{L}_3=0$.

Put $\bar{L}_4=\tau(\bar{L}_3)$ and $\bar{L}_5=\tau(\bar{L}_2)$. Then $\bar{C}\cdot\bar{L}_4=\bar{C}\cdot\bar{L}_5=0$ and

$$-K_{\bar{X}}\cdot\bar{L}_4=-K_{\bar{X}}\cdot\bar{L}_5=E_4\cdot\bar{L}_4=E_5\cdot\bar{L}_5=1,$$

which implies that $E_i\cdot\bar{L}_5=E_j\cdot\bar{L}_4=0$ for every $i\neq 5$ and $j\neq 4$.

Put $L_3=\pi(\bar{L}_3)$, $L_4=\pi(\bar{L}_4)$, $L_2=\pi(\bar{L}_2)$ and $L_5=\pi(\bar{L}_5)$. Then

$$L_3+L_4\sim L_2+L_5\sim -2K_X,$$

which implies that $\mathrm{c}(X,L_3+L_4)=1/3$ and $\mathrm{c}(X,L_2+L_5)=1/2$. Then $\mathrm{lct}_2(X)\leqslant 2/3$. But

$$\bar{L}_2\sim_{\mathbb{Q}}\pi^*\big(L_2\big)-\frac{5}{7}E_1-\frac{10}{7}E_2-\frac{8}{7}E_3-\frac{6}{7}E_4-\frac{4}{7}E_5-\frac{2}{7}E_6-\frac{1}{2}F,$$

$$\bar{L}_3\sim_{\mathbb{Q}}\pi^*\big(L_3\big)-\frac{4}{7}E_1-\frac{8}{7}E_2-\frac{12}{7}E_3-\frac{9}{7}E_4-\frac{6}{7}E_5-\frac{3}{7}E_6,$$

which implies that $\mathrm{c}(X,2L_2+L_3)=1/4$. Then $2L_2+L_3\sim_{\mathbb{Q}}-3K_X$, since $\mathrm{Pic}(X)\cong\mathbb{Z}^2$ and

$$L_2\cdot L_2=\frac{3}{7},\ L_3\cdot L_3=\frac{5}{7},\ L_2\cdot L_3=\frac{8}{7},$$

but $2L_2+L_3$ is a Cartier divisor, which implies that $2L_2+L_3\sim -3K_X$.

If D is not a curve in $|-K_X|$ and $D\neq (L_3+L_4)/2$, then arguing as in the proof of Lemma 4.11, we easily see that $\mu>2/3$, since we can use (6.4). The lemma is proved (see Example 1.27). $\square$

Lemma 6.8. Suppose that $m=5$. Then $\mu\geqslant\mathrm{lct}_2(X)=2/3$, and if $\mu=2/3$, then

- either D is a curve in $|-K_X|$ with a cusp at a point in $\mathrm{Sing}(X)$ of type $\mathbb{A}_2$,
- or the divisor D is uniquely defined and can be explicitly described.

Proof. The curve R' has an ordinary tacnodal singularity at the point $\omega'(P')$, which implies that there exists a line $L'\subset\mathbb{P}^2$ such that either $L'\subset\mathrm{Supp}(R')$ or $L'\not\subset\mathrm{Supp}(R')$ and

$$\mathrm{mult}_{\omega'(P')}\Big(L'\cdot R'\Big)=4.$$

There are irreducible smooth rational curves L_3' and L_4' on the surface X' such that

$$\omega'\big(L_3'\big)=\omega'\big(L_4'\big)=L'$$

and $L_3'=L_4'\iff L'\subset\mathrm{Supp}(R')$. Note that neither L_3' nor L_4' contains a point in $\mathrm{Sing}(X')\backslash R'$.

Let $\bar{L}_3'$ be the proper transform of the curve L_3' on the surface $\bar{X}'$. Then

$$\bar{L}_3'\cap\eta\big(E_1\big)=\bar{L}_3'\cap\eta\big(E_2\big)=\bar{L}_3'\cap\eta\big(E_4\big)=\bar{L}_3'\cap\eta\big(E_5\big)=\varnothing,$$

and $\bar{L}_3'\cdot\eta(E_3)=1$. Let $\bar{L}_4'$ be the proper transform of the curve L_4' on the surface $\bar{X}'$. Then

$$\bar{L}_4'\cap\eta\big(E_1\big)=\bar{L}_4'\cap\eta\big(E_2\big)=\bar{L}_4'\cap\eta\big(E_4\big)=\bar{L}_4'\cap\eta\big(E_5\big)=\varnothing,$$

and $\bar{L}_4'\cdot\eta(E_3)=1$. One can also check that $\bar{L}_3'\cap\bar{L}_4'=\varnothing$ if $\bar{L}_3'\neq\bar{L}_4'$.

Let $\bar{L}_3$ and $\bar{L}_4$ be the proper transforms of the curves $\bar{L}_3'$ and $\bar{L}_4'$ on the surface $\bar{X}$, respectively, and let us put $L_3=\pi(\bar{L}_3)$ and $L_4=\pi(\bar{L}_4)$. Then

$$\bar{L}_3+\bar{L}_4\sim -2K_X$$

and $\mathrm{c}(X,\bar{L}_3+\bar{L}_4)=1/3$, which implies that $\mathrm{lct}_2(X)\leqslant 2/3$.

If $D\neq(\bar{L}_3+\bar{L}_4)/2$, then (6.4), the proof of Lemma 4.10 and Lemma 2.6 imply that

$$\mu\geqslant\mathrm{lct}_2\big(X\big)=\frac{2}{3}.$$

and if $\mu = 2/3$, then D is a curve in $|-K_X|$ with a cusp at a point in $\mathrm{Sing}(X)$ of type $\mathbb{A}_2$. □

Lemma 6.9. Suppose that $m = 4$. Then

$$\mu \geqslant \mathrm{lct}_2\big(X\big) = \min\big(\mathrm{lct}_1\big(X\big), 4/5\big) \geqslant \frac{2}{3},$$

and if $\mu = 2/3$, then D is a curve in $|-K_X|$ with a cusp at a point in $\mathrm{Sing}(X)$ of type $\mathbb{A}_2$.

Proof. The point $\omega'(P')$ is an ordinary cusp of the curve R'. Then there is a line $L' \subset \mathbb{P}^2$ such that

$$\mathrm{mult}_{\omega'(P')}\Big(L' \cdot R'\Big) = 3.$$

Let Z' be a curve in X' such that $\omega'(Z') = L'$ and $-K_{X'} \cdot Z' = 2$. Then

$$Z' \cap \mathrm{Sing}\big(X'\big) = \mathrm{Sing}\big(Z'\big) = R',$$

the Z' is irreducible curve that has an ordinary cusp at the point R'.

Let $\bar{Z}'$ be the proper transform of the curve Z' on the surface $\bar{X}'$. Then Z' is smooth and

$$\eta\big(E_2\big) \cap \eta\big(E_3\big) \in \bar{Z}'.$$

Let $\bar{Z}$ be the proper transform of the curve $\bar{Z}'$ on the surface $\bar{X}$. Put $Z = \pi(\bar{Z})$. Then

$$\bar{Z} \sim \pi^*\big(Z\big) - E_1 - 2E_2 - 2E_3 - E_4$$

and $E_2 \cap E_3 \in Z$. Then $\mathrm{c}(X, Z) = 2/5$, which implies that $\mathrm{lct}_2(X) \leqslant 4/5$.

Arguing as in the proof of Lemma 4.8 and using Lemma 2.6 and (6.4), we see that

$$\mu \geqslant \mathrm{lct}_2\big(X\big) = \min\big(\mathrm{lct}_1\big(X\big), 4/5\big)$$

and if $\mu = 2/3$, then D is a curve in $|-K_X|$ with a cusp at a point in $\mathrm{Sing}(X)$ of type $\mathbb{A}_2$. □

The assertion of Theorem 6.1 is proved.

References

[1] V. Alexeev, V. Nikulin, *Del Pezzo and K3 Surfaces*
MSJ Memoirs **15** (2006), Mathematical Society of Japan, Tokyo

[2] I. Cheltsov, *Log canonical thresholds on hypersurfaces*
Sbornik: Mathematics **192** (2001), 1241–1257

[3] I. Cheltsov, *Log canonical thresholds of del Pezzo surfaces*
Geometric and Functional Analysis, **18** (2008), 1118–1144

[4] I. Cheltsov, *On singular cubic surfaces*
Asian Journal of Mathematics **19** (2009), 191–214

[5] I. Cheltsov, J. Park, J. Won, *Log canonical thresholds of certain Fano hypersurfaces*
arXiv:0706.0751 (2007)

[6] I. Cheltsov, C. Shramov, *Log canonical thresholds of smooth Fano threefolds*
Russian Mathematical Surveys **63** (2008), 859-958

[7] I. Cheltsov, C. Shramov, *On exceptional quotient singularities*
Geometry and Topology **15** (2011), 1843–1882

[8] X. Chen, B. Wang, *Kähler–Ricci flow on Fano manifolds*
arXiv:0909.2391 (2009)

[9] F. Cools, M. Coppens, *Star points on smooth hypersurfaces*
Journal of Algebra **323** (2010), 261–286

[10] J.-P. Demailly, J. Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*
Annales Scientifiques de l'École Normale Supérieure **34** (2001), 525–556

[11] W. Ding, G. Tian, *Kähler-Einstein metrics and the generalized Futaki invariant*
Inventiones Mathematicae, **110** (1992), 315–335

[12] S. Donaldson, *Stability, birational transformations and the Kahler-Einstein problem* arXiv:1007.4220 (2010)

[13] T. de Fernex, L. Ein, M. Mustata, *Bounds for log canonical thresholds with applications to birational rigidity* Mathematical Research Letters **10** (2003), 219–236

[14] A. Futaki, *An obstruction to the existence of Einstein–Kähler metrics* Inventiones Mathematicae **73** (1983), 437–443

[15] J. Gauntlett, D. Martelli, J. Sparks, S.-T. Yau, *Obstructions to the existence of Sasaki-Einstein metrics* Communications in Mathematical Physics **273** (2007) 803–827

[16] A. Ghigi, J. Kollár,*Kähler-Einstein metrics on orbifolds and Einstein metrics on spheres* Commentarii Mathematici Helvetici, **82** (2007), 877–902

[17] P. Hacking, Y. Prokhorov, *Smoothable del Pezzo surfaces with quotient singularities* Compositio Mathematica **146** (2010), 169–192

[18] F. Hidaka, K. Watanabe, *Normal Gorenstein surfaces with ample anti-canonical divisor* Tokyo Journal of Mathematics **4** (1981), 319-330

[19] T. Jeffres, *Singular set of some Kähler orbifolds* Transactions of the American Mathematical Society **349** (1997), 1961–1971

[20] S. Keel, J. McKernan, *Rational curves on quasi-projective surfaces* Memoirs of the American Mathematical Society **669** (1999), American Mathematical Society

[21] J. Kollár, *Singularities of pairs* Proceedings of Symposia in Pure Mathematics **62** (1997), 221–287

[22] J. Kollár et al., *Flips and abundance for algebraic threefolds* Astérisque **211** (1992)

[23] G. Lyubeznik, *On Bernstein–Sato polynomials* Proceedings of the American Mathematical Society **125** (1997), 1941–1944

[24] T. Mabuchi, S. Mukai, *Stability and Einstein–Kähler metric of a quartic del Pezzo surface* Lecture Notes in Pure and Applied Mathematics **145** (1993), 133–160

[25] Y. Matsushima, *Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne* Nagoya Mathematical Journal **11** (1957), 145–150

[26] M. Miyanishi, D.-Q. Zhang, *Gorenstein log del Pezzo surfaces of rank one* Journal of Algebra **118** (1988), 63–84

[27] M. Mustata, *Singularities of pairs via jet schemes* Journal of the American Mathematical Society **15** (2002), 599–615

[28] J. Park, *Birational maps of Del Pezzo fibrations* Journal fur die Reine und Angewandte Mathematik **538** (2001), 213–221

[29] J. Park, *A note on del Pezzo fibrations of degree* 1 Communications in Algebra, **31** (2003), 5755–5768

[30] J. Park, J. Won *Log canonical thresholds on Gorenstein canonical del Pezzo surfaces* Proceedings of the Edinburgh Mathematical Society, **54** (2011), 187–219

[31] J. Park, J. Won *Log canonical thresholds on del Pezzo surfaces of degree* $\geqslant 2$ Nagoya Mathematical Journal, **200** (2010), 1–26

[32] Yu. Prokhorov, *Blow-ups of canonical singularities* arXiv:math/9810097 (1998)

[33] A. Pukhlikov, *Birational geometry of Fano direct products* Izvestiya: Mathematics **69** (2005), 1225–1255

[34] J. Ross, R. Thomas, *Weighted projective embeddings, stability of orbifolds and constant scalar curvature Kahler metrics* arXiv:math/09075214 (2009)

[35] Y. Shi, *On the α-invariants of cubic surfaces with Eckardt points* arXiv:0902.3203 (2009)

[36] G. Tian, *On Kähler–Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$* Inventiones Mathematicae **89** (1987), 225–246

[37] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class* Inventiones Mathematicae, **101** (1990), 101–172

[38] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds* Journal of Differential Geometry **32** (1990), 99–130

[39] G. Tian, *Kähler-Einstein metrics with positive scalar curvature* Inventiones Mathematicae **130** (1997), 1–37

[40] T. Urabe, *On singularities on degenerate Del Pezzo surfaces of degree* 1*,* 2 Proceedings of Symposia in Pure Mathematics **40** (1983), 587–590

[41] B. Wang, *Ricci flow on orbifolds* arXiv:1003.0151 (2010)

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